Rewriting methods in higher algebra

Yves Guiraud

Université Paris 7
Habilitation à diriger des recherches
Spécialité : Mathématiques

Méthodes de réécriture en algèbre supérieure

Yves Guiraud

Jury

Pierre-Louis CURIEN	CNRS, Université Paris 7	Rapporteur
Graham ELLIS	National University of Ireland, Galway	Examinateur
Meinolf GECK	Universität Stuttgart	Examinateur
Gérard HUET	INRIA	Examinateur
Muriel LIVERNET	Université Paris 7	Examinatrice
Georges MALTSINIOTIS	CNRS, Université Paris 7	Président
Martin MARKL	Institute of Mathematics CAS	Rapporteur
Ieke MOERDIJK	Universiteit Utrecht	Rapporteur

Soutenue le 18 juin 2019

Rewriting methods in higher algebra

Yves Guiraud

November 7, 2018
Last modified on June 20, 2019

Contents

Chapter 1
Introduction

1.1. General presentation

This manuscript is a summary of my research activity, with an emphasis on the last ten years. My mathematical interests lie mainly in algebra, with a particular focus on the following subjects:

(i) *Combinatorial algebra*, and more precisely all the mathematical descriptions of what is an algebraic structure, such as operads of all kinds, props and variations, or Lawvere theories. Here, my main motivation lies in the utopia of a universal algebraic object subsuming all these concepts, yet keeping an easy access to the combinatorics, but also including higher-dimensional algebraic structures, and formalisms from proof theory or theoretical computer science, like the sequent calculus, rewriting systems, or the λ-calculus.

(ii) *Homotopical algebra*, and model structures in particular, as a unified point of view on the concept of topology-inspired equivalence between mathematical objects, including homological algebra as a special case. In this vast domain, I am particularly interested in cofibrant approximations, as an abstract description of the concept of resolution, with the underlying idea that cofibrant approximations in well-chosen model categories should provide good candidates to formalise mathematical structures in logical languages such as type theory.

(iii) *Higher-dimensional algebra*, whose central objective is the description and classification of all the species of higher categories, strict or weak, globular or cubical, invertible or not, with or without an algebraic structure. I would also include concepts from homological algebra, like chain complexes and differential graded algebras and categories, and from homotopical algebra, like cellular complexes. Here, coherence problems, for all sorts of relaxed categorical objects ranging from pseudofunctors to weak higher groupoids, have a special appeal to me.

(iv) *Effective algebra*, also known as computer algebra or symbolic computation, concerned with the formalisation of algebraic structures and the mechanisation of algebraic computation. Here, I have particularly worked on presentations by generators and relations of monoids, categories and higher categories, with a specific interest into rewriting theory, both to understand the algebraic structure underlying the rewriting mechanism, and to develop rewriting methods for various applications: computation of normal forms and linear bases, decision of the word problem, computation of resolutions and of homotopical and homological invariants, resolution of coherence problems.

Among these algebraic subjects, the major part of my work has been devoted to the interactions between rewriting theory and higher categories, with a shift of focus roughly ten years ago.

During the first period, ranging from 2000 to 2007 and covering my PhD thesis and postdoc years, I was interested into understanding rewriting theory from an algebraic point of view. This led me to the development of a general rewriting theory in higher-dimensional categories, based on Burroni's structure of polygraph. I have explored translations of various types of rewriting-like systems into polygraphs, and the respective computational properties of the two formalisms, for word and term rewriting systems, Petri nets, and deep-inference proofs of propositional logic and linear logic. Using algebraic derivations, I have developed a method to prove termination of polygraphs; this was later, with Guillaume Bonfante, specialised to the complexity analysis of a subclass of polygraphs corresponding to first-order functional programs, leading to a new characterisation of the FP complexity class. All these results are, in essence, natural extensions of my PhD thesis, and, for that reason, in this manuscript, I chose to focus on later work; however, all the concepts of polygraphic rewriting, being essential in what followed, are still presented here, in a version that has progressively been modernised all along the years.

The second period opens in 2008, when Philippe Malbos and I started to explore a higher-dimensional formulation of Squier's work on homological and homotopical invariants of rewriting. This led us into the development of rewriting methods for the computation of various constructions in homotopical algebra: syzygies and coherence bases, homological and homotopical invariants, resolutions by higher categories. Our different articles were motivated by various applications, such as the refinement of Squier's algebraic invariants of rewriting, constructive proofs of coherence theorems in categorical algebra, the computation of polygraphic resolutions of monoids, the study of homotopical properties of Artin monoids (with Stéphane Gaussent), the implementation of resulting algorithms (with Samuel Mimram), or the characterisation of the Koszulness property for associative algebras (with Eric Hoffbeck). In the last years, following the study of Artin monoids, in joint works with Patrick Dehornoy and Matthieu Picantin, I have started to explore the interactions between rewriting and Garside theory, motivated by possible enhancements that the latter can bring to the former.

The next two sections of this introduction propose a brief survey on the rewriting theory from an algebraic point of view, and on Squier's work on homological and homotopical invariants of rewriting. We end the chapter with a summary of this work.

1.2. Rewriting theory in algebra

1.2.1. The word problem. The word problem for finitely presented monoids originates in 1914, when Thue asked in [199] if, given a finite presentation $X = (X_1, X_2)$ of a monoid by generators and relations, there exists an algorithm to decide whether or not any two given elements u and v of the free monoid X_1^* over X_1 are equal up to the relations of X_2. In 1947, Post [179] and Markov [157] simultaneously and independently answered by the negative. Their proof is based on an encoding of a Turing machine as a presentation of a monoid, in such a way that the decidability of the word problem in this presentation implies the one of another, undecidable problem: namely, if the machine will ever write a given symbol or not. Still in 1947, Markov gave a more concrete counterexample: a monoid presented by 13 generators and 33 relations, with an undecidable word problem [158]. In 1956 and 1958, Tseitin proposed smaller examples of monoids with undecidable word problems, both admitting rather simple presentations with five generators and seven relations [202, 203, 204].

1.2.2. Rewriting theory. On the positive side, one way to solve the word problem is to exhibit a finite presentation $X = (X_1, X_2)$ of M, made of a generating set X_1 and a set X_2 of *directed* relations with a good computational property: *convergence*. The use of directed relations is the central concept of rewriting theory, where one studies presentations whose relations are not seen as equalities between the words in X_1^*, such as $u = v$, but, instead, as *rewriting rules* that can only be applied in one direction, like $u \Rightarrow v$, thus simulating a nonreversible computational process reducing the word u into the word v. A presentation X is called *convergent* if it has the two properties of

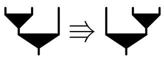
(i) *termination*, i.e. all the computations end eventually, and

(ii) *confluence*, i.e. different computations on the same input eventually lead to the same result.

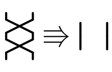
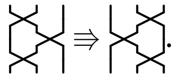
A finite and convergent presentation X of a monoid M gives a solution to the word problem for M, called the normal-form procedure and defined as follows. Given an element u of the free monoid X_1^*, convergence ensures that all the applications of (directed) relations to u, in every possible manner, will eventually produce a unique result: an element $\widehat{u}$ of X_1^* where no relation applies anymore. The word $\widehat{u}$ is called the *normal form of* u, and the map $u \mapsto \widehat{u}$ induces a (set-theoretic) section of the canonical projection $X_1^* \twoheadrightarrow M$: by construction, two elements u and v of X_1^* represent the same element of M if, and only if, their normal forms $\widehat{u}$ and $\widehat{v}$ are equal in the free monoid X_1^*. Finiteness ensures that one can determine whether an element u of X_1^* is a normal form or not, by examining each relation to check if it applies to u or not. Note that finiteness is just a sufficient condition here, since other methods may exist to check whether or not u is a normal form.

In 1943, Newman gave a general setting, abstract rewriting, to describe the properties of termination and confluence, and to show the first fundamental result of rewriting: Newman's lemma [169] asserts that, under the termination hypothesis, confluence can be reduced to a local form, that only checks if one-step rewritings acting on the same element eventually lead to the same result. Since then, rewriting theory has been mainly and independently developed in effective algebra and theoretical computer science, resulting in numerous variants corresponding to different syntaxes of the elements being transformed, such as word or string rewriting systems [32] for monoids, or term rewriting systems [10, 125, 198] for equational theories and Lawvere theories [140]. Other computational systems are also based on rewriting-like mechanisms, like Gröbner bases in effective algebra [42], or the λ-calculus [54] and Petri nets [168] in theoretical computer science.

1.2.3. Polygraphs. More recently, higher-dimensional rewriting has unified several paradigms of rewriting. This theory is based on presentations by generators and relations of higher categories, independently introduced by Burroni and Street under the respective names of *polygraphs* in [43] and *computads* in [195, 196]. Burroni's terminology is more widely used in the French community, partly due to our closeness to him, and partly due to aesthetic considerations. Polygraphs are a higher-categorical analogue of CW complexes: they are formed by cells of all dimensions, and each cell has the shape of a globe, whose boundary lives in the free higher category over the cells of lower dimension. For example, the presentation of the monoid $A = \langle a \mid aa = a \rangle^{+}$ is a 2-polygraph with one 0-cell, one 1-cell a, and one 2-cell $aa \Rightarrow a$. Similarly, the presentation of the associative theory is a 3-polygraph with one 0-cell, one 1-cell, one 2-cell standing for the product, and pictured by  in the formalism of string diagrams, and one 3-cell



expressing the associativity of the product. Another example is the categorical presentation of the groups of permutations: it has one 2-cell , standing for the generating transposition $(1\ 2)$, and the following two 3-cells, respectively expressing that  is an involution and that it satisfies the Yang-Baxter relation:

 and 

The fundamental virtue of polygraphs is that they encapsulate, in the same globular object, the terms, the computations on the terms, and the homotopical properties of the computations. Thus, polygraphs are perfectly adequate to study higher-dimensional algebraic structures, either from a combinatorial and computational point of view [43, 133, 135, 88, 90, 89, 92, 91, 136, 93, 30, 31, 164], or for homological or homotopical reasons [160, 161, 137, 138, 9]. Moreover, polygraphs provide a natural setting to relate these two different faces of higher algebras, and, in particular, to formulate Squier theory, based on the discovery of deep relations between the homological and homotopical properties of algebraic objects, on the one hand, and the combinatorial and computational properties of their presentations, on the other hand.

1.3. An overview of Squier theory

1.3.1. Jantzen's question, and Kapur-Narendran's example. The normal-form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties, leading Jantzen to ask if every finitely presented monoid with a decidable word problem admits a finite convergent presentation [115, 116]. The answer was expected to be negative, but the proof was revealed to be even harder than expected by the following observation. In [122], Kapur and Narendran consider Artin's presentation of the monoid B_3^+ of positive braids on three strands:

$$(s, t \mid sts \Rightarrow tst)\,.$$

Since B_3^+ has a finite presentation with homogeneous relations, it has a solvable word problem: the equivalence class of a given word u on s and t can be totally explored to check if another word v belongs to it or not. Kapur and Narendran proved that B_3^+ admits no finite convergent presentation on the two generators s and t. However, they also proved that B_3^+ admits a finite convergent presentation, if one adjoins to s and t an extra, redundant generator a standing for the product st:

$$\left(s, t, a \;\middle|\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \right). \tag{1.1}$$

Hence, the existence of a finite convergent presentation depends on the chosen generators. It follows that, to give the awaited negative answer to Jantzen's question, one would have to exhibit a monoid with a decidable word problem but with no finite convergent presentation *for any possible set of generators*: new methods had to be introduced to tackle the problem.

1.3.2. From computational to homological properties. And, indeed, Craig Squier answered the question by linking the existence of a finite convergent presentation for a given monoid M to a homological invariant of M: a monoid M is *left*-FP$_3$ if there exists an exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3$$

of projective and finitely generated $\mathbb{Z}M$-modules, where $\mathbb{Z}$ denotes the trivial $\mathbb{Z}M$-module. From a presentation X of M, one can build an exact sequence of free $\mathbb{Z}M$-modules

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \varepsilon\ } \mathbb{Z}M \xleftarrow{\ d_1\ } \mathbb{Z}M[X_1] \xleftarrow{\ d_2\ } \mathbb{Z}M[X_2], \tag{1.2}$$

where $\mathbb{Z}M[X_k]$ is the free $\mathbb{Z}M$-module over X_k. So, if the kernel of d_2 is finitely generated, then M is left-FP$_3$. Moreover, by classical arguments of homological algebra, the fact that the kernel of d_2 is finitely generated or not does not depend on the choice of a *finite* presentation of M.

In [192], Squier proves that, if X is convergent, its *critical branchings* form a generating set of the kernel of d_2, where a critical branching of X is a minimal overlapping application of two relations on the same element of X_1^*. For example, the relations $\alpha : ta \Rightarrow as$ and $\beta : st \Rightarrow a$ of (1.1) generate a critical branching $(\beta a, s\alpha)$ on sta:

$$\begin{tikzcd}[row sep=small]
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas
\end{tikzcd}$$

The convergence of (1.1) ensures that every critical branching (f, g) is confluent, which means that it can be completed by rewriting sequences f' and g' leading to the same result, as in

$$\begin{tikzcd}[row sep=small]
 & v \arrow[dr, Rightarrow, bend left, "f'"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & u' \\
 & w \arrow[ur, Rightarrow, bend right, "g'"'] &
\end{tikzcd} \tag{1.3}$$

For example, the presentation (1.1) of B_3^+ has four critical branchings, and all of them are confluent:

$$\begin{tikzcd}[row sep=small]
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas \arrow[uu, Rightarrow, "\gamma"']
\end{tikzcd}
\quad
\begin{tikzcd}[row sep=small]
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
 & saa \arrow[uu, Rightarrow, "\delta"']
\end{tikzcd}
\quad
\begin{tikzcd}[row sep=small]
 & aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
 & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[row sep=small]
 & aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, bend left, "\gamma aa"] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & & \\
 & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}$$

Squier proved that the set X_3 of critical branchings of a convergent presentation X extends the exact sequence (1.2) by one step:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \varepsilon\ } \mathbb{Z}M \xleftarrow{\ d_1\ } \mathbb{Z}M[X_1] \xleftarrow{\ d_2\ } \mathbb{Z}M[X_2] \xleftarrow{\ d_3\ } \mathbb{Z}M[X_3], \qquad (1.4)$$

where the boundary map d_3 is defined on the generic branching (1.3) by

$$d_3(f,g) \;=\; [f] - [g] + [f'] - [g'],$$

Since the set of critical branchings of a finite convergent presentation is itself finite, we obtain that, if a monoid admits a finite convergent presentation, then it is left-FP_3 [192, Theorem 4.1]. Finally, Squier considers a family $(S_k)_{k\geqslant 2}$ of monoids, such that each S_k is finitely generated and has a decidable word problem, but, if $k \geqslant 2$, then S_k is not left-FP_3 and, as a consequence, it does not admit a finite convergent presentation (for any possible set of generators).

1.3.3. From computational to homotopical properties. Later, Squier gave another proof of the same result, based on a homotopical refinement of the property left-FP_3. Given a monoid M with a presentation X, Squier considers in his posthumous article [193] the complex of the presentation X: this is a cellular complex with one 0-cell for each element of X_1^*, and whose 1-cells correspond to one-step rewritings. Then, he extends this 1-dimensional complex with 2-cells filling all the squares formed by independent applications of relations, such as the following one, provided there exist 1-cells between u_1 and v_1, and between u_2 and v_2:

$$\begin{tikzcd}[column sep=small]
 & wv_1w'u_2w'' \arrow[dr, dash] & \\
wu_1w'u_2w'' \arrow[ur, dash] \arrow[dr, dash] & & wv_1w'v_2w'' \\
 & wu_1w'v_2w'' \arrow[ur, dash] &
\end{tikzcd}$$

Squier defines a *homotopy basis of* X as a generating set of the fundamental group of the resulting two-dimensional complex, and proves that, if X and Y are two finite presentations of the same monoid, then X admits a finite homotopy basis if, and only if, Y does [193, Theorem 4.3]. Next, Squier defines the following homotopical invariant for monoids: one says that a monoid is *of finite derivation type* if it admits a finite presentation with a finite homotopy basis. He then proves that, if X is a convergent presentation of a monoid, and if Y is a set of 2-cells filling the diagram (1.3) for each critical branchings of X, then Y is a homotopy basis of X [193, Theorem 5.2]. As a consequence, if a monoid admits a finite convergent presentation, then it is of finite derivation type [193, Theorem 5.3]. It was later proved that, if a monoid is of finite derivation type, then it is left-FP_3 [57, 181, 133].

Squier uses these results to give another proof that there exist finitely generated monoids, with a decidable word problem, but that do not admit a finite convergent presentation. Moreover, he shows that left-FP_3 is not sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation. Indeed, the monoid S_1 of [192], given by the finite presentation $(a,b,t,x,y \mid ab \Rightarrow 1,\ xa \Rightarrow atx,\ xt \Rightarrow tx,\ xb \Rightarrow bx,\ xy \Rightarrow 1)$, has a decidable word problem, it is left-FP_3, yet it is not of finite derivation type, and, as a consequence, it does not admit a finite convergent presentation.

1.4. Organisation and notation

1.4.1. Organisation. Chapter 2 gives the general definitions, notations and constructions used throughout this document: higher categories, polygraphs and their rewriting properties. The material is collected from all my articles, with a presentation that has evolved all along the years. Here, I tried to present the concepts and constructions in the most recent form. In particular, higher categories are defined in their internal version, useful to adapt them from monoids to other algebraic structures as in [95]. The rewriting terminology comes from the same article. The method to prove termination originates in [88, 90, 89], and is formulated in the language of [96]. This last article is also the source of the classification of critical branchings in dimension three.

Chapter 3 summarises the results of [96], [97] and [98], written with Philippe Malbos. The first article was devoted to a generalisation of Squier's homotopical theorem to higher categories, with a special interest for strict monoidal categories; the theorem was expressed as a rewriting-based construction of a coherent presentation, i.e. a presentation by generators, relations, and relations among relations. The second article first adapted to higher categories the notion of identity among relations, known for presentations of groups as a way to express the syzygies formed by the relations, and then used Squier's theorem to compute generators of the identities among relations. The third article used higher categories to give simple formulations of some categorical coherence problems, and applied Squier's theorem to obtain new proofs of the coherence theorems for monoidal, symmetric monoidal and braided monoidal categories.

Chapter 4 recalls the results of [99], written with Philippe Malbos. The main objective of this article was to extend to higher-dimensions Squier's homotopical theorem for monoids and categories, obtaining the construction of a polygraphic resolution from a convergent presentation: polygraphic resolutions are resolutions by higher categories, extending coherent presentations in all dimensions. Moreover, we used polygraphic resolutions to define various homotopical invariants, and, through an abelianisation process, homological invariants as well, generalising Squier's finite homotopy type and left-FP_3 conditions. Here, the construction of the polygraphic resolution is presented with slightly less restrictive hypotheses than in the original article, and with a corrected proof, the original one containing an error in the definition of some coherence cells in dimension 4 and above.

Chapter 5 presents the article [78], written with Stéphane Gaussent and Philippe Malbos. Our objective was to understand two results about different homotopical aspects of Artin monoids, from the point of view of Squier theory. The first one, by Tits, concerns the fundamental group of a complex associated to the classical presentation of Artin monoids. The second result, by Deligne, is a characterisation of the weak actions of spherical Artin monoids on categories, in terms of another presentation of Artin monoids whose generators are the elements of the corresponding Coxeter group. The main result we obtained was an enhanced version of the construction given by Squier's theorem, generating a rather compact coherent presentation from a non-necessarily terminating presentation. Applied to Artin monoids, this construction gave two different coherent presentations, which, in turn, induced improvements of Tits' and Deligne's results.

Chapter 6 summarises the results of [63], written with Patrick Dehornoy. This article introduced an axiomatic setting, quadratic normalisations, to encompass two methods to compute normal forms in monoids: on the one hand, convergent quadratic polygraphs (such as the convergent Garside presentation of Artin monoids, or the column presentations of plactic and Chinese monoids), and, on the other hand, the normalisation process induced by a Garside family. Quadratic normalisations are classified according to their class (m, n), determined by the maximal length of reductions on length-three words: m when starting from the left, and n from the right. We particularly explored the quadratic normalisations of class $(4, 3)$, proving that they induce convergent quadratic presentations of monoids, that they generalise the Garside case, and we gave a characterisation of the Garside normalisations among them.

Chapter 7 presents the results of [95], written with Eric Hoffbeck and Philippe Malbos. Our objective was to transpose Squier theory to associative algebras, in order to give a rewriting account of the known computational methods using Gröbner bases to compute normal forms, bases, and resolutions of associative algebras. For that, we developed a rewriting theory for associative algebras, based on an analogue of polygraphs for higher associative algebras, i.e. higher categories internal to the category of associative algebras. This rewriting theory resulted into a new concept of convergent presentations for algebras, that strictly generalise noncommutative Gröbner bases and Poincaré-Birkhoff-Witt bases, and into Squier-like constructions of coherent presentations and of polygraphic resolutions of associative algebras, leading to new sufficient or necessary conditions for an algebra to be Koszul.

Finally, Chapter 8 contains research perspectives, beyond the results presented in this manuscript, ranging from almost completed work to long-term ideas, in four directions: the general theory of higher categories and polygraphs, to extend the range of application of rewriting methods to new algebraic structures; the study of the interactions between Garside theory and rewriting, both to improve rewriting itself, and Squier theory as a consequence; the computation and study of polygraphic resolutions of Artin monoids and groups, with a view towards the $K(\pi, 1)$-conjecture; and the development of new algebraic invariants of computation, related to the word problem or to complexity theory.

1.4.2. Notation. In all the document, we denote by $\overline{\mathbb{N}}$ the set $\mathbb{N}$ of natural numbers extended with a maximal element ∞. If M is a monoid, we write M_+ for the semigroup obtained from M by removing the unit. If X is a set, then X^* denotes the the free monoid over X. So, for example, $\mathbb{N}_+$ is the set of nonzero natural numbers, $\mathbb{N}^*$ the free monoid over $\mathbb{N}$, and $(\mathbb{N}_+)^*$ the free monoid over nonzero natural numbers. Specific notation is also introduced in the last chapters.

Chapter 2
Higher categories and polygraphs

2.1. Introduction

2.1.1. Context. The structure of higher category is implicitly already present in classical, 1-dimensional categories: setting aside size considerations, categories, functors and natural transformations form a 2-category, where categories are 0-dimensional objects, or 0-cells, functors are 1-cells, with one way to compose them, and natural transformations are 2-cells, being composable along two possible dimensions [154]. In turn, 2-categories, 2-functors, (pseudo)natural transformations and modifications between them yield a 3-category, while 3-categories with their morphisms of all dimensions (3-functors, natural transformations, modifications, and perturbations) form a 4-category, and so on. At infinity, one obtains the (large) ∞-category of all (small) ∞-categories. More precisely, these are the strict, globular ∞-categories, in that all the laws satisfied by compositions are strict equalities, and all n-cells, for $n \geqslant 1$, have one $(n-1)$-cell as source, and one $(n-1)$-cell as target, with the shape of a globe.

This precise type of higher categories is now rather well-understood, and the category $\infty\mathcal{C}at$ they form has a nice model structure [138]. Over the years, many other types of higher categories have also been studied, coming initially from algebraic topology and mathematical physics, and, more recently, from proof theory [111, 182]. Without trying to be exhaustive, either in the types of higher categories or in the now vast bibliography on the subject, higher categories can have different shapes of cells, such as cubical or opetopic cells [38, 16]; the cells can be invertible or not [154, 18]; the compositions can satisfy relations that are hold strictly, or only up to coherent higher cells [23, 83, 118, 156]; the cells can admit an extra algebraic structure, such as an associative product or a Lie bracket, and the laws of this structure can themselves hold strictly or only up to coherent higher cells [124]. Comparisons and equivalences have been established, for example between various definitions of globular higher categories [53], or between globular and cubical higher groupoids and categories [39, 38, 2].

Polygraphs, also known as computads, were introduced independently by Street [195, 196, 197] and Burroni [43] as a combinatorial algebra point of view on (strict, globular) higher categories: polygraphs are to higher categories what generating families, presentations by generators and relations, and resolutions are to monoids and groups. More precisely, polygraphs are a common algebraic description of sets of generating cells for free higher categories, and of presentations by generators and relations for higher categories that are free up to codimension 1. Later on, Métayer has developed a rich notion of polygraphic resolution for higher categories [160], and proved that polygraphs are cofibrant for an adequate family of cofibrations [161]. These first homotopical observations integrated later into a model structure on $\infty\mathcal{C}at$ [138]. A general description for polygraphs/computads for various types of higher categories was given by Batanin in [17].

The higher-categorical interpretation of rewriting originates in Burroni's original article [43], where he gave a polygraphic description of Lawvere theories [140] as specific 2-categories, and presentations of Lawvere theories as specific 3-polygraphs. Since polygraphs have globular cells, with a source and a target, they can be seen naturally as directed presentations of higher categories, and Lafont started to explore their rewriting properties from this observation [133, 135]. In my PhD thesis and subsequent articles, I gave a general setting for higher-dimensional rewriting, showing how to translate other types of rewriting-like mechanisms in this language: word and term rewriting systems, directed presentations of pros, props and Lawvere theories, Petri nets, and propositional logic and linear logic in their deep inference formulation [88, 89, 92, 91]. Specific tools have been developed to study the rewriting properties of polygraphs, especially in dimension 3: for example, algebraic derivations to prove termination [90, 89], leading with Bonfante to the study of the implicit complexity of polygraphs and to polygraphic characterisations of some complexity classes [30, 31]; or completion procedures to compute convergent presentations of pros [164, 165].

2.1.2. Summary. In §2.2, we give a definition of higher categories internal to a category $\mathcal{C}$ with good properties (having pullbacks, in particular), as in [95]. This relies on the notion of globular object in $\mathcal{C}$, defined as a graded object with source s, target t and identity maps i that satisfy the so-called globular relations: $ss = st$, $ts = tt$ and $si = ti = \mathrm{id}$. The general underlying idea is that an n-category is an n-globular set equipped with n compatible structures of monoids, one for each possible dimension of composition. An important concept is the one of n-sphere in a higher category, defined as a pair of n-cells that are parallel, i.e. that share the same source and the same target. We are also interested in the following variant: (n, p)-categories are n-categories whose k-cells are invertible for every $k > p$.

Then, §2.3 focuses on the case of n-categories in the category *Set* of sets, or just n-categories for short. The central concept of this section is that of extension of an n-category $\mathcal{C}$, which is a distinguished set of n-spheres of $\mathcal{C}$. An extension X of $\mathcal{C}$ can be seen as a set of formal $(n+1)$-cells, filling all the n-spheres of X: taking all formal compositions of these $(n+1)$-cells yields an $(n+1)$-category, denoted by $\mathcal{C}[X]$; an analogue construction, where one also considers formal inverses of cells of X gives an $(n+1, p)$-category $\mathcal{C}(X)$ from an (n, p)-category $\mathcal{C}$. An extension X of $\mathcal{C}$ can also be seen as a set of relations between parallel n-cells: collapsing them yields a quotient n-category, written $\mathcal{C}/X$. From [96], we also define the contexts of an n-category $\mathcal{C}$, as n-cells with one formal indeterminate, and natural systems on $\mathcal{C}$ as the functors from the category $\mathrm{Ct}(\mathcal{C})$ of contexts of $\mathcal{C}$ into a semiadditive category, such as the one of abelian groups. The category of contexts of $\mathcal{C}$ has been introduced by Quillen under the name category of factorisations of $\mathcal{C}$ in [183], and it has been used by Leech to introduce cohomological properties of congruences on monoids in [141], and by Baues and Wirsching for the cohomology of small categories in [20]. Here, natural systems are used as a notion of homological coefficients for higher categories, generalising left/right/bimodules of categories.

Next, §2.4 recalls the definition of polygraphs from [43, 160], which are a higher categorical analogue of CW complexes: a polygraph X is a sequence $(X_0 \,|\, \cdots \,|\, X_n \,|\, \cdots)$ where X_0 is a set, and each X_{n+1} is an extension of the free n-category X_n^* generated by the n-cells. The main difference with globular sets is that the source and target of a generating $(n+1)$-cell are not necessarily generating n-cells, but can be any composite. We also give a variant of polygraphs for (n,p)-categories, as introduced in [99]: these (n,p)-polygraphs are defined similarly, except that, above dimension p, one considers extensions of the free (n,p)-category $X_n^\top$ generated by the n-cells. Polygraphs provide a notion of presentation for higher categories that are free up to codimension 1, together with generalised versions in higher dimensions: in particular, if $\mathcal{C}$ is an n-category that is free up to codimension 1, a coherent presentation of $\mathcal{C}$ is an $(n+2,n)$-polygraph formed by a presentation X of $\mathcal{C}$ together with an acyclic extension of $X^\top$.

Then, §2.5 recalls the essential aspects of the rewriting theory of n-polygraphs, as developed progressively in [88, 89, 96]. The elementary concept is that of rewriting step, which is an n-cell that contains only one n-dimensional generator, seen as a transformation of its source into its target. Termination is then classically defined as the fact that no infinite sequence of composable rewriting steps exist, and confluence as the fact that any two finite sequence of rewriting steps with the same source can be completed into sequences with the same target. An n-polygraph is then called convergent if it is both terminating and confluent. The two fundamental results of rewriting theory hold in the case of polygraphs:

Theorem 2.5.4. *Fix $n > 0$, and let X be an n-polygraph.*

- ***(i)*** (Newman's lemma) *If X terminates, then X is confluent if, and only if, it is locally confluent.*
- ***(ii)*** (The critical branchings theorem) *X is locally confluent if, and only if, it is critically confluent.*

The proofs are postponed until Chapter 3, where more general versions are given: Propositions 3.2.2 and 3.2.3.

Finally, §2.6 summarises results from [96] on the rewriting properties of 2-polygraphs and 3-polygraphs. In dimension 2, we recall a basic link between the decidability of the word problems for monoids and categories, and the existence of a finite convergent presentation, and we cite simple facts about termination and confluence of 2-polygraphs. For dimension 3, we give the method to prove termination of 3-polygraphs by using derivations of 2-categories, initially introduced in [88, 89], in the formulation of [96]. We conclude with the analysis of the specific critical branchings that one encounters in dimension 3, recalling their classification from [96] and the results we obtained there.

2.2. Internal higher categories

Fix $n \in \overline{\mathbb{N}}$, and let $\mathcal{C}$ be a fixed category. The definitions of n-globular objects of $\mathcal{C}$ and n-categories of $\mathcal{C}$ can be given in a more abstract setting, but we assume here that $\mathcal{C}$ is concrete over sets, and that the corresponding forgetful functor admits a left adjoint.

2.2.1. Graded objects and morphisms. An *n-graded object of $\mathcal{C}$* is a sequence $X = (X_k)_k$ of objects of $\mathcal{C}$, indexed by all natural numbers $k \leqslant n$. If X and Y are n-graded objects of $\mathcal{C}$, and p an integer, a *graded morphism of degree p from X to Y* is a sequence

$$f = \big(X_k \to Y_{k+p}\big)_k$$

of morphisms of $\mathcal{C}$, indexed by all natural numbers $k \leqslant n$ such that $0 \leqslant k + p \leqslant n$. The n-graded objects and graded morphisms of degree 0 of $\mathcal{C}$ form a category, which has limits and colimits, computed pointwise, if $\mathcal{C}$ has.

2.2.2. Internal globular objects. An n-*globular object of* $\mathcal{C}$ is an n-graded object X of $\mathcal{C}$ equipped with graded morphisms

$$X \xrightarrow{s} X, \qquad X \xrightarrow{t} X \qquad \text{and} \qquad X \xrightarrow{i} X,$$

of respective degrees -1, -1 and 1, called the *source map*, the *target map* and the *identity map of* X, that satisfy the following equalities, collectively referred to as the *globular relations of* X:

$$ss = st, \qquad ts = tt \qquad \text{and} \qquad si = ti = \mathrm{id}_X\,.$$

With the identity map and the last two relations removed, one gets an n-*semiglobular object of* $\mathcal{C}$, or n-*graph of* $\mathcal{C}$. Usually, "globular objects" correspond to our semiglobular objects, while "reflexive globular objects" stand for our "globular objects"; the present terminology is chosen here for its coherence with (semi)simplicial and (semi)cubical objects.

Given two n-globular objects X and Y of $\mathcal{C}$, a *globular morphism from* X *to* Y is a graded morphism $f : X \to Y$ of degree 0 that commutes with the source, target and identity maps. We denote by $n\mathcal{Glob}(\mathcal{C})$ the category of n-globular objects and globular morphisms of $\mathcal{C}$, and by $n\mathcal{Gph}(\mathcal{C})$ the category of n-graphs and their morphisms.

2.2.3. Sources, targets, spheres and compositions. Let X be an n-globular object of $\mathcal{C}$, and fix a natural number $k \leqslant n$. An element x of X_k is called a k-*cell of* X, and, if $k \geqslant 1$, the $(k-1)$-cells $s(x)$ and $t(x)$ are called the *source of* x and the *target of* x. We write $z : x \to y$ if z is a k-cell of X of source x and target y, and $z : x \Rightarrow y$, $z : x \Rrightarrow y$, $z : x \mathrel{\equiv\!\!\!>} y$ if $k = 2, 3, 4$. When no confusion occurs, we write $i(x)$, 1_x or just x instead of any iterate image $i^p(x)$ of x through i. Two k-cells x and y of X are called *parallel* if either $k = 0$, or $s(x) = s(y)$ and $t(x) = t(y)$. A k-*sphere of* X is a pair $\alpha = (x, y)$ of parallel k-cells of X, with x being called the *source of* α and y the *target of* α. The *boundary* of a k-cell x is the $(k-1)$-sphere $\partial(x) = (s(x), t(x))$.

Writing (X_k) for the n-globular object of $\mathcal{C}$ that is constantly equal to X_k, the k-*source map of* X and the k-*target map of* X are the graded morphisms $s_k, t_k : X \to (X_k)$ of degree 0, given, on a p-cell x of X, by

$$s_k(x) = \begin{cases} s^{p-k}(x) & \text{if } p \geqslant k, \\ i^{k-p}(x) & \text{if } p \leqslant k, \end{cases} \qquad \text{and} \qquad t_k(x) = \begin{cases} t^{p-k}(x) & \text{if } p \geqslant k, \\ i^{k-p}(x) & \text{if } p \leqslant k. \end{cases}$$

The globular relations generalise, for $j < k$, to $s_j s_k = s_j t_k = s_j$ and $t_j s_k = t_j t_k = t_j$ for $j < k$. We denote by $X \star_k X$ the pullback

$$\begin{tikzcd} X \star_k X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "s_k"] \\ X \arrow[r, "t_k"'] & (X_k) \end{tikzcd}$$

in the category of n-graded objects of $\mathcal{C}$. Explicitly, the p-cells of $X \star_k X$ are the pairs (x, y) of p-cells of X such that $t_k(x) = s_k(y)$ holds, such a pair being called k-*composable*. By definition of s_k and t_k, if $p \leqslant k$, then the p-cells of $X \star_k X$ are the (x, x), for every p-cell x of X.

2.2.4. Internal higher categories. An n-*category of* $\mathcal{C}$ is an n-globular object $\mathcal{C}$ of $\mathcal{C}$ equipped, for every natural number $k < n$, with a graded morphism

$$\mathcal{C} \star_k \mathcal{C} \xrightarrow{c_k} \mathcal{C}$$

of degree 0, called the k-*composition of* $\mathcal{C}$, whose value at (a, b) is denoted by $a \star_k b$, and such that the following relations are satisfied, for all $0 \leqslant k < p$:

(i) (*compatibility with the source and target maps*) for every p-cell (a, b) of $\mathcal{C} \star_k \mathcal{C}$,

$$s(a \star_k b) = \begin{cases} s(a) & \text{if } k = p-1, \\ s(a) \star_k s(b) & \text{otherwise,} \end{cases}$$

$$t(a \star_k b) = \begin{cases} t(b) & \text{if } k = p-1, \\ t(a) \star_k t(b) & \text{otherwise,} \end{cases}$$

(ii) (*compatibility with the identity map*) for every p-cell (a, b) of $\mathcal{C} \star_k \mathcal{C}$,

$$1_{a \star_k b} = 1_a \star_k 1_b,$$

(iii) (*associativity*) for all p-cells a, b and c of $\mathcal{C}$ such that (a, b) and (b, c) are p-cells of $\mathcal{C} \star_k \mathcal{C}$,

$$(a \star_k b) \star_k c = a \star_k (b \star_k c),$$

(iv) (*neutrality*) for every p-cell a of $\mathcal{C}$,

$$s_k(a) \star_k a = a = a \star_k t_k(a),$$

(v) (*exchange*) for every $j < k$, and all p-cells (a, a') and (b, b') of $\mathcal{C} \star_k \mathcal{C}$ such that (a, b) and (a', b') are p-cells of $\mathcal{C} \star_j \mathcal{C}$,

$$(a \star_k a') \star_j (b \star_k b') = (a \star_j b) \star_k (a' \star_j b').$$

Note that the compatibility of the compositions with the source and target maps ensures that the associativity axiom makes sense: if (a, b) and (b, c) are p-cells of $\mathcal{C} \star_k \mathcal{C}$, then so do $(a \star_k b, c)$ and $(a, b \star_k c)$. The compatibility of compositions with identities implies that we can still write a for 1_a with no ambiguity. We use the convention that the composition $\star_j$ binds more tightly than $\star_k$ for $j < k$, so $a \star_j b \star_k c$ means $(a \star_j b) \star_k c$.

Given n-categories $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{C}$, an n-*functor of* $\mathcal{C}$ *from* $\mathcal{C}$ *to* $\mathcal{D}$ is a globular morphism F from $\mathcal{C}$ to $\mathcal{D}$ that commutes with all k-compositions. We denote by $n\mathcal{C}at(\mathcal{C})$ the category of n-categories and n-functors of $\mathcal{C}$.

2.2.5. Internal (n, p)-categories. In an n-category $\mathcal{C}$ of $\mathcal{C}$, for $n \geqslant 1$, an n-cell a is called *invertible* if there exists an n-cell a^- in $\mathcal{C}$, of source $t(a)$ and target $s(a)$, such that the relations

$$a \star_{n-1} a^- = s(a) \qquad \text{and} \qquad a^- \star_{n-1} a = t(a)$$

are satisfied. For $p \leqslant n$ in $\overline{\mathbb{N}}$, an (n, p)-*category of* $\mathcal{C}$ is an n-category of $\mathcal{C}$ in which all k-cells are invertible for $k > p$. In particular, (n, n)-categories are just n-categories, and we say n-*groupoids* for $(n, 0)$-categories. We denote by $(n, p)\mathcal{C}at(\mathcal{C})$ and $n\mathcal{G}pd(\mathcal{C})$ the corresponding categories.

2.3. Higher categories

2.3.1. Notation. When $\mathcal{C}$ is the category of sets, we speak of n-graded sets, n-globular sets, n-categories, etc. and denote by $n\mathcal{Glob}$, $n\mathcal{Cat}$, etc. the corresponding categories. All these categories are known to be complete and cocomplete. For $0 \leqslant p < n \leqslant \infty$, the two adjunctions

$$\begin{tikzcd} p\mathcal{Cat} \arrow[r, bend left, "I^n"] \arrow[r, phantom, "\perp"] & n\mathcal{Cat} \arrow[l, bend left, "U_p"] \arrow[r, bend left, "T_p"] \arrow[r, phantom, "\perp"] & (n,p)\mathcal{Cat} \arrow[l, bend left] \end{tikzcd}$$

are formed by: the *truncation* functor U_p, that forgets the cells above dimension p; the *inclusion* functor I^n, that sees a p-category as an n-category with only identity cells above dimension p; the *enveloping* (n,p)*-category* functor T_p, adding formal inverses for all cells above dimension p; the functor that forgets inverses. Through the inclusion functor, a p-category $\mathcal{C}$ is implicitly seen as an n-category, also denoted by $\mathcal{C}$. If $\mathcal{C}$ is an n-category, we abusively write $\mathcal{C}_p$ for its underlying p-category $U_p(\mathcal{C})$. Finally, we write $\mathcal{C}^\top$ for $T_p(\mathcal{C})$ when there is no ambiguity on p.

2.3.2. Standard cells and spheres. Fix a natural number n. The *standard* n*-cell* is the n-category $\mathcal{E}^n$ with two k-cells e_k^- and e_k^+ for every $k < n$, and one n-cell e_n, plus all corresponding identities; the source and target maps of $\mathcal{E}^n$ are given by $s_k(e) = e_k^-$ and $t_k(e) = e_k^+$ for every cell e of $\mathcal{E}^n$. With identities removed, the first three standard cells $\mathcal{E}^0$, $\mathcal{E}^1$ and $\mathcal{E}^2$ are

$$e_0 \qquad\qquad e_0^- \xrightarrow{e_1} e_0^+ \qquad\qquad \begin{tikzcd} e_0^- \arrow[r, bend left, "e_1^-"] \arrow[r, bend right, "e_1^+"'] \arrow[r, phantom, "\Downarrow e_2"] & e_0^+ \end{tikzcd}$$

The *standard* n*-sphere* is the n-category $\mathcal{S}^n = U(\mathcal{E}^{n+1})$. By extension, we define the *standard* (-1)*-sphere* as $\mathcal{S}^{-1} = \emptyset$.

Let $\mathcal{C}$ be an n-category, for $n \in \overline{\mathbb{N}}$, and $k \leqslant n$ be a natural number. By definition of $\mathcal{E}^k$ and $\mathcal{S}^k$, the k-cells and the k-spheres of $\mathcal{C}$ are in bijective correspondence with the n-functors $\mathcal{E}^k \to \mathcal{C}$ and $\mathcal{S}^k \to \mathcal{C}$, respectively. In what follows, we use these two points of view on cells and spheres of $\mathcal{C}$ indistinctly.

2.3.3. Extensions. Fix a natural number n, and an n-category $\mathcal{C}$. An *extension of* $\mathcal{C}$ is a pair (X, φ) made of a set X and an n-functor

$$X \times \mathcal{S}^n \xrightarrow{\varphi} \mathcal{C}.$$

By definition, the image of φ is a set of n-spheres of $\mathcal{C}$, indexed by X. We usually identify each element x of X with the corresponding n-sphere $\varphi(x, (e_{n-1}^-, e_{n-1}^+))$ of $\mathcal{C}$, and leave the notation φ implicit, just saying that X is an extension of $\mathcal{C}$. An n-category $\mathcal{C}$ admits two canonical extensions: the empty one, and the extension $n\,\mathrm{Sph}(\mathcal{C})$ formed by all the n-spheres of $\mathcal{C}$.

Let X be an extension of $\mathcal{C}$. We denote by $\approx_X$ the smallest equivalence relation on parallel n-cells of X that is compatible with compositions, and contains all the n-spheres $(s(x), t(x))$ for x in X. We say that X is *acyclic* if $a \approx_X b$ holds for every n-sphere (a, b) of $\mathcal{C}$.

We denote by $n\mathcal{C}at^+$ the category of pairs $(\mathcal{C}, X)$ made of an n-category and an extension X of $\mathcal{C}$, obtained as the pullback of the forgetful functor $n\mathcal{C}at \to n\mathcal{G}ph$ and the truncation functor $(n+1)\mathcal{G}ph \to n\mathcal{G}ph$:

$$\begin{tikzcd} n\mathcal{C}at^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (n+1)\mathcal{G}ph \arrow[d] \\ n\mathcal{C}at \arrow[r] & n\mathcal{G}ph. \end{tikzcd}$$

For $p \leqslant n$, we denote by $(n, p)\mathcal{C}at^+$ the subcategory of $n\mathcal{C}at^+$ formed by (n, p)-categories with an extension.

2.3.4. Adjoining and collapsing cells. Fix a natural number n, an n-category $\mathcal{C}$, and an extension X of $\mathcal{C}$. We define $\mathcal{C}[X]$ and $\mathcal{C}/X$ as the $(n+1)$-category and the n-category given by the following pushouts in $(n+1)\mathcal{C}at$ and $n\mathcal{C}at$:

$$\begin{tikzcd} X \times \mathbb{S}^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ X \times \mathcal{E}^{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{C}[X] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times \mathbb{S}^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ X \times \mathcal{E}^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{C}/X. \end{tikzcd}$$

These two constructions extend to functors $n\mathcal{C}at^+ \to (n+1)\mathcal{C}at$ and $n\mathcal{C}at^+ \to n\mathcal{C}at$, respectively. Concretely, $\mathcal{C}[X]$ is obtained from $\mathcal{C}$ by adjoining the composites of formal $(n+1)$-cells, one for each n-sphere of X, while the n-category $\mathcal{C}/X$ is obtained by collapsing all the n-spheres of X. We denote by $\pi_X : \mathcal{C} \twoheadrightarrow \mathcal{C}/X$ the canonical projection. In practice, we use the same notation for an n-sphere of X and the corresponding $(n+1)$-cell of $\mathcal{C}[X]$. We write $\|a\|_X$ for the number of occurrences of cells of X in an n-cell a of $\mathcal{C}[X]$, and $\mathrm{Cell}_X(a)$ for the set of cells of X that appear in a.

By construction, for every n-sphere (a, b) of $\mathcal{C}$, the following assertions are equivalent:

(i) $a \approx_X b$,

(ii) $\pi_X(a) = \pi_X(b)$,

(iii) there exists a zigzag of $(n+1)$-cells from a to b in $\mathcal{C}[X]$,

(iv) there exists an $(n+1)$-cell from a to b in $\mathcal{C}(X)$.

For $p \leqslant n$, and $\mathcal{C}$ an (n, p)-category, we define the $(n+1, p)$-category $\mathcal{C}(X)$ is the same way as $\mathcal{C}[X]$, but taking the pushout in $(n+1, p)\mathcal{C}at$. The construction also applies to an n-category $\mathcal{C}$, in the case $p = n$, yielding an $(n+1, n)$-category $\mathcal{C}(X)$. Concretely, $\mathcal{C}(X)$ is obtained from $\mathcal{C}$ by formally adjoining the composites of the n-spheres of X and their inverses. The notations $\|a\|_X$ and $\mathrm{Cell}_X(a)$ are extended to $\mathcal{C}(X)$ by considering the minimal number and minimal set of cells of X that appear in a.

2.3.5. Contexts and whiskers. Let n be a natural number, and $\mathcal{C}$ be an n-category. The construction $\mathcal{C}[X]$ is extended in a straightforward way to a set of k-spheres of $\mathcal{C}$ for any $k \leqslant n$, and so is the notation $\|a\|_X$.

A *context of* $\mathcal{C}$ is a pair (x, C), written $C[x]$, made of an $(n-1)$-sphere x of $\mathcal{C}$ and an n-cell C of $\mathcal{C}[x]$ such that $\|C\|_x = 1$. By construction of $\mathcal{C}[x]$, for every context $C[x]$ of $\mathcal{C}$, the n-cell C admits a decomposition

$$C = a_n \star_{n-1} \cdots \star_1 a_1 \star_0 x \star_0 b_1 \star_1 \cdots \star_{n-1} b_n,$$

where a_k and b_k are k-cells of $\mathcal{C}$, for every k in $\{1, \dots, n\}$. If there exists such a decomposition where both a_n and b_n are identities, then $C[x]$ is called a *whisker of* $\mathcal{C}$.

If $C[x]$ is a context of $\mathcal{C}$, and a is an n-cell of $\mathcal{C}[X]$ such that $\partial a = x$, for X an extension of $\mathcal{C}$, we define the n-cell $C[a]$ of $\mathcal{C}[X]$ as C where the n-cell x has been replaced by a. In particular, if $D[y]$ is a context of $\mathcal{C}$ such that $\partial D = x$, the n-cell $C[D]$ contains exactly one occurrence of y, so that $(y, C[D])$ is a context of $\mathcal{C}$, denoted by $C[D[y]]$.

The contexts of $\mathcal{C}$ form a category, denoted by $\mathrm{Ct}(\mathcal{C})$, whose objects are the n-cells of $\mathcal{C}$, and whose morphisms from a to b are the contexts $C[\partial a]$ of $\mathcal{C}$ such that $C[a] = b$ holds. Composition is given by $D[y] \circ C[x] = D[C[x]]$ and the identity of a is the *empty* context $[\partial a] = (\partial a, \partial a)$.

2.3.6. Natural systems and derivations. Let n be a natural number, and $\mathcal{C}$ be an n-category. A *natural system on* $\mathcal{C}$ is a functor from the category of contexts $\mathrm{Ct}(\mathcal{C})$ of $\mathcal{C}$ to the category $\mathcal{Ab}$ of abelian groups. Hence, a natural system N is specified by an abelian group N_a, for every n-cell a of $\mathcal{C}$, and a morphism $N_{C[x]} : N_a \to N_{C[a]}$ of groups, for every context $C[x]$ of $\mathcal{C}$ and every n-cell a such that $\partial(a) = x$. When no confusion may occur, if $C[x] : a \to b$ is a morphism of $\mathrm{Ct}(\mathcal{C})$, and g is an element of N_a, we denote by $C[g]$ the element $N_{C[x]}(g)$ of N_b. The category of natural systems over $\mathcal{C}$ is denoted by $\mathcal{Nat}(\mathcal{C})$.

Let N be a natural system on $\mathcal{C}$. A *derivation of* $\mathcal{C}$ *into* N is a map sending every n-cell a of $\mathcal{C}$ to an element $d(a)$ of N_a such that the following relation holds, for every k-composable pair (a, b) of n-cells of $\mathcal{C}$:

$$d(a \star_k b) = a \star_k d(b) + d(a) \star_k b.$$

A direct consequence of this relation is that derivations map identities to 0.

Natural systems on $\mathcal{C}$ and the associated derivations can also be defined with values in any semiadditive category $\mathcal{A}$ instead of $\mathcal{Ab}$, and in particular in the category $\mathcal{Com}$ of commutative monoids. In that case, we talk about natural systems on $\mathcal{C}$ *with values in* $\mathcal{A}$.

2.3.7. Examples. For example, the *trivial* natural system on $\mathcal{C}$, denoted by $\mathbb{Z}$, maps every n-cell of $\mathcal{C}$ to $\mathbb{Z}$ and every context of $\mathcal{C}$ to the identity of $\mathbb{Z}$. If X is an extension of $\mathcal{C}$, the map $\|\cdot\|_X$, counting the number of occurrences of n-cells of X in the n-cells of $\mathcal{C}[X]$, is a derivation of $\mathcal{C}$ into $\mathbb{Z}$, mapping every n-cell of $\mathcal{C}$ to 0 and every n-cell of X to 1. The trivial natural system on $\mathcal{C}$ with values in $\mathcal{Com}$ is defined similarly, with $\mathbb{N}$ replacing $\mathbb{Z}$.

Let $\mathcal{V}$ a concrete category, usually the category $\mathcal{Set}$ of sets or the category $\mathcal{Ord}$ of ordered sets. We view $\mathcal{V}$ as a 2-category with one 0-cell and cartesian product as 0-composition. Fix an internal commutative monoid M in $\mathcal{V}$, a 2-category $\mathcal{C}$ and 2-functors $F : \mathcal{C} \to \mathcal{V}$ and $G : \mathcal{C}^{co} \to \mathcal{V}$, where $\mathcal{C}^{co}$ is $\mathcal{C}$ with the 1-source and 1-target maps exchanged. The following assignments yield a natural system $N(F, G, M)$ over $\mathcal{C}$ with values in $\mathcal{Com}$:

(i) Every 2-cell $f : a \Rightarrow b$ is mapped to the set $N(F, G, M)_f$ of morphisms from $F(a) \times G(b)$ to M in $\mathcal{V}$, equipped with the pointwise structure of commutative monoid induced by the one of M.

(ii) For every 1-sphere (a, b) and 1-cell c of $\mathcal{C}$, the values of $N(F, G, M)$ on the contexts $(a, b) \star_0 c$ and $c \star_0 (a, b)$ are given, on $\varphi : F(a) \times G(b) \to M$ in $\mathcal{V}$, by

$$\begin{array}{rcl} F(a) \times F(c) \times G(b) \times G(c) & \xrightarrow{\varphi \star_0 c} & M \\ (\xi, \xi', \eta, \eta') & \longmapsto & \varphi(\xi, \eta), \end{array}$$

$$\begin{array}{rcl} F(c) \times F(a) \times G(c) \times G(b) & \xrightarrow{c \star_0 \varphi} & M \\ (\xi', \xi, \eta', \eta) & \longmapsto & \varphi(\xi, \eta). \end{array}$$

(iii) For every 1-sphere (a, b) and 2-cells $f : a' \to a$ and $g : b \to b'$ of $\mathcal{C}$, the values of $N(F, G, M)$ on the contexts $f \star_1 (a, b)$ and $(a, b) \star_1 g$ are given, on $\varphi : F(a) \times G(b) \to M$ in $\mathcal{V}$, by

$$\begin{array}{rcl} F(a') \times G(b) & \xrightarrow{f \star_1 \varphi} & M \\ (\xi, \eta) & \longmapsto & \varphi\big(F(f)(\xi), \eta\big), \end{array}$$

$$\begin{array}{rcl} F(a) \times G(b') & \xrightarrow{\varphi \star_1 g} & M \\ (\xi, \eta) & \longmapsto & \varphi\big(\xi, G(g)(\eta)\big). \end{array}$$

If F or G is trivial, i.e. maps all the cells of $\mathcal{C}$ to the terminal object $*$ of $\mathcal{V}$, one denotes the corresponding natural system by $N(*, G, M)$ or $N(F, *, M)$. In particular, $N(*, *, \mathbb{N})$ is the trivial natural system on $\mathcal{C}$ with values in $\mathcal{C}om$.

2.4. Polygraphs

2.4.1. Polygraphs. We define the category $n\mathcal{P}ol$ of *n-polygraphs* and the *free n-category functor* $n\mathcal{P}ol \to n\mathcal{C}at$ by induction on $n \geqslant 0$. For $n = 0$, put $0\mathcal{P}ol = \mathcal{S}et$, and set the free 0-category functor to the identity of $\mathcal{S}et$, which is well typed because $0\mathcal{C}at = \mathcal{S}et$. Now, fix $n \geqslant 1$, and assume that the category $(n-1)\mathcal{P}ol$ and the functor $(n-1)\mathcal{P}ol \to (n-1)\mathcal{C}at$ have been defined. The category $n\mathcal{P}ol$ is defined as the pullback

$$\begin{tikzcd} n\mathcal{P}ol \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (n-1)\mathcal{C}at^{+} \arrow[d] \\ (n-1)\mathcal{P}ol \arrow[r] & (n-1)\mathcal{C}at \end{tikzcd} \tag{2.1}$$

and the free n-category functor, as the composite

$$n\mathcal{P}ol \longrightarrow (n-1)\mathcal{C}at^{+} \longrightarrow n\mathcal{C}at.$$

If X is an n-polygraph, we denote by X^* the free n-category over X. Expanding the definition, an n-polygraph is a family $X = (X_0, \ldots, X_n)$, written $(X_0 \,|\, \cdots \,|\, X_n)$, made of a set X_0 and, for every $0 \leqslant k < n$, an extension X_{k+1} of the free k-category over the k-polygraph $(X_0, \ldots, X_k)$. The free n-category over X is $X^* = X_0[X_1] \cdots [X_n]$.

The category $\infty\mathcal{P}ol$ of ∞-*polygraphs* is obtained as the limit of the vertical functors $(n+1)\mathcal{P}ol \to n\mathcal{P}ol$ of (2.1), and the *free ∞-category functor* comes from the universal property of $\infty\mathcal{C}at$ as a limit of the functors $(n+1)\mathcal{C}at \to n\mathcal{C}at$. Thus, an ∞-polygraph is a sequence $X = (X_0 \,|\, \cdots \,|\, X_n \,|\, \cdots)$ such that $(X_0 \,|\, \cdots \,|\, X_n)$ is an n-polygraph for every $n \geqslant 0$.

Fix $n \in \overline{\mathbb{N}}$, let X be an n-polygraph, and let $k \leqslant n$ be a natural number. The elements of X_k are called the k-*cells of* X. We commit the abuse to also denote by X_k the underlying k-polygraph of X. We say that X is of *finite type* if it has finitely many k-cells for every $k \geqslant 0$. For an n-cell a of X^*, we put $\|a\| = \|a\|_{X_n}$, calling it the *size of* a, and $\mathrm{Cell}(a) = \mathrm{Cell}_{X_n}(a)$.

2.4.2. Polygraphs for (n,p)-categories. We define the category $(n,p)\mathcal{P}ol$ of (n,p)-*polygraphs*, together with the *free (n,p)-category functor* $(n,p)\mathcal{P}ol \to (n,p)\mathcal{C}at$, for all natural numbers $n \geqslant p$, by induction on $n-p$. For $n = p$, put $(n,n)\mathcal{P}ol = n\mathcal{P}ol$, and set the free (n,n)-category functor to the free n-category functor, which is well typed because $(n,n)\mathcal{C}at = n\mathcal{C}at$. Now, fix $n > p$, and assume that the category $(n-1,p)\mathcal{P}ol$ and the functor $(n-1,p)\mathcal{P}ol \to (n-1,p)\mathcal{C}at$ have been defined. The category $(n,p)\mathcal{P}ol$ is defined as the pullback

$$\begin{tikzcd}
(n,p)\mathcal{P}ol \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (n-1,p)\mathcal{C}at^{+} \arrow[d] \\
(n-1,p)\mathcal{P}ol \arrow[r] & (n-1,p)\mathcal{C}at
\end{tikzcd}$$

and the free (n,p)-category functor, as the composite

$$(n,p)\mathcal{P}ol \longrightarrow (n-1,p)\mathcal{C}at^{+} \longrightarrow (n,p)\mathcal{C}at.$$

If X is an (n,p)-polygraph, we denote by $X^{\top}$ the free (n,p)-category over X. Thus, an (n,p)-polygraph is a family $X = (X_0 \,|\, \cdots \,|\, X_n)$ such that $(X_0 \,|\, \cdots \,|\, X_p)$ is a p-polygraph, and each X_k, for $k > p$, is an extension of the free $(k-1,p)$-category over $(X_0 \,|\, \cdots \,|\, X_{k-1})$. The free (n,p)-category over X is $X^{\top} = X_0[X_1]\cdots[X_p](X_{p+1})\cdots(X_n)$. Note that $(n,n-1)$-polygraphs and n-polygraphs have the same definition, giving a meaning to $X^{\top}$ for an n-polygraph X.

Finally, we define the category $(\infty,p)\mathcal{P}ol$ of (∞,p)-*polygraphs* as the limit of the functors $(n+1,p)\mathcal{P}ol \to (n,p)\mathcal{P}ol$. We use the same vocabulary as in the case of ∞-polygraphs.

2.4.3. Notation. To help to differentiate the various types of cells that appear in (free) higher categories, we use the following convention as much as possible.

If $\mathcal{C}$ is a generic higher category, we write a, b, c and f, g, h for its cells, the latter being of higher dimension with respect to the former. If X is a generic polygraph, we use x, y, z for its cells.

When focusing on the lowest dimensions, such as when considering a free $(\infty,1)$-category $X^{\top}$, we use x, y, z for the 1-cells of X, and u, v, w for the 1-cells of $X^{\top}$; for $n > 1$, we write α, β, γ for the n-cells of X, and a, b, c for the n-cells of $X^{\top}$. Moreover, in that case, we drop the notation $\star_0$ for the 0-composition of cells in $X^{\top}$, so that uv stands for the 0-composite of 1-cells u and v of $X^{\top}$.

2.4.4. Polygraphic presentations. Fix $n \in \overline{\mathbb{N}}$, and let $\mathcal{C}$ be an n-category. We say that $\mathcal{C}$ is *free* if there exists an n-polygraph X such that $\mathcal{C} \simeq X^*$; in that case, the cells of X are called the *generating cells of* $\mathcal{C}$. For $k < n$, we say that $\mathcal{C}$ is k-*free* if its underlying k-category $\mathcal{C}_k$ is free.

Let $p < n$, and X be an (n,p)-polygraph. The p-*category presented by* X is the quotient p-category

$$\overline{X} = X_p^*/X_{p+1}.$$

We denote by $\overline{a}$ the image of a p-cell a of X_p^* in $\overline{X}$ through the canonical projection. If f is a k-cell of $X^\top$, for $k > p$, we also denote by $\overline{f}$ the common image in $\overline{X}$ of the p-cells $s_p(f)$ and $t_p(f)$ through the canonical projection. Two n-polygraphs, seen as $(n, n-1)$-polygraphs, are called *Tietze-equivalent* if the $(n-1)$-categories they present are isomorphic.

Assume that n is finite, and let $\mathcal{C}$ be an n-category. A *presentation of* $\mathcal{C}$ is an $(n+1)$-polygraph X such that $\mathcal{C} \simeq \overline{X}$; in that case, we say that $\mathcal{C}$ is *presented by* X. If $\mathcal{C}$ admits a finite presentation, then it is *finitely presented*. This definition is restrictive: for $n \geqslant 1$, only the $(n-1)$-free n-categories can have a presentation. A *coherent presentation of* $\mathcal{C}$ is an $(n+2, n)$-polygraph X whose presented n-category is isomorphic to $\mathcal{C}$, and such that X_{n+2} is acyclic.

2.4.5. Example (The standard presentation). Let n be a natural number, and $\mathcal{C}$ be an $(n-1)$-free n-category. The *standard presentation of* $\mathcal{C}$ is the $(n+1)$-polygraph $\mathrm{Std}(\mathcal{C})$ with the following cells:

- **(i)** up to dimension $n-1$, one cell for each generating cell of $\mathcal{C}$,
- **(ii)** one n-cell $\widehat{a}$ for every n-cell a of $\mathcal{C}$, with the same source and target,
- **(iii)** one $(n+1)$-cell $\gamma_k(a,b) : \widehat{a} \star_k \widehat{b} \to \widehat{a \star_k b}$ for every k-composable pair (a,b) of n-cells of $\mathcal{C}$, and one $(n+1)$-cell $\iota(a) : 1_a \to \widehat{1}_a$ for every $(n-1)$-cell a of $\mathcal{C}$, with the following sources and targets:

$$\begin{tikzcd} & t_k(a) = s_k(b) \arrow[dr, "\widehat{b}"] \arrow[d, phantom, "\Downarrow \gamma^k_{a,b}"] & \\ s_k(a) \arrow[ur, "\widehat{a}"] \arrow[rr, "\widehat{a \star_k b}"'] & & t_k(b) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right, "\widehat{1}_a"'] \arrow[r, phantom, "\Downarrow \iota_a"] & a. \end{tikzcd}$$

2.5. Rewriting properties of polygraphs

We fix a natural number $n > 0$ and an n-polygraph X for the whole section.

2.5.1. Rewriting steps and normal forms. A *rewriting step of* X is an n-cell of size 1 in the free n-category X^*. A *rewriting sequence of* X is a finite or infinite sequence

$$a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} (\cdots) \xrightarrow{f_{n-1}} a_n \xrightarrow{f_n} (\cdots)$$

of composable rewriting steps of X. Note that every n-cell of X^* decomposes into a finite rewriting sequence of X, and this decomposition is unique up to the exchange relations. If a and b are $(n-1)$-cells of X^*, we say that a *rewrites into* b if there exists a finite rewriting sequence of X with source a and target b.

Let a be an $(n-1)$-cell of X^*. We say that a is *reduced* if there exists no rewriting step of source a in X. A *normal form of* a is a reduced $(n-1)$-cell b of X^* such that a rewrites into b. We say that X is *normalising* if every n-cell of X^* admits at least one normal form.

2.5.2. Termination. Let a be an $(n-1)$-cell of X^*. We say that X *terminates at* a if X has no infinite rewriting sequence of source a. By definition, if X terminates at a, then a admits a normal form. We say that X *terminates* (or is *terminating*) if it does at every $(n-1)$-cell of X^*. Thus, if X terminates, it is normalising. Moreover, if X terminates, then putting $a \succcurlyeq_X b$ for all $(n-1)$-cells a and b of X^* such that a rewrites into b defines a well-founded order on the $(n-1)$-cells of X^*. In that case, induction on $\succcurlyeq_X$ is called *noetherian induction*.

A *termination order on* X is a wellfounded order relation $\leqslant$ on parallel $(n-1)$-cells of X^* such that the following properties are satisfied:

(i) the compositions of $(n-1)$-cells of X^* are strictly monotone in both arguments,

(ii) for every n-cell x of X, the strict inequality $s(x) > t(x)$ holds.

As a direct consequence of the definition, if X admits a termination order, then X terminates. The converse is also true, because, if X terminates, then $\succcurlyeq_X$ is a termination order.

2.5.3. Branchings and confluence. A *branching of* X is a non-ordered pair (f, g) of n-cells of X^* with same source, as in

$$b \xleftarrow{\;f\;} a \xrightarrow{\;g\;} c\,;$$

in that case, a is called the *source of* (f, g), and we say that (f, g) is *local* if f and g are rewriting steps of X. A local branching of X is called *trivial* if it has one of the following two shapes

$$a' \xleftarrow{\;f\;} a \xrightarrow{\;f\;} a' \qquad \text{or} \qquad a' \star_k b \xleftarrow{\;f \star_k b\;} a \star_k b \xrightarrow{\;a \star_k g\;} a \star_k b',$$

where $f : a \to a'$ and $g : b \to b'$ are rewriting steps. Local branchings are compared by inclusion, i.e. by the order $\preccurlyeq$ generated by the relations

$$(f, g) \preccurlyeq (C[f], C[g])$$

for every possible whisker $C[x]$ of X^*. A non-trivial local branching of X that is minimal for the order $\preccurlyeq$ is called *critical*. By induction on the size of the source of a local branching, we deduce that, for every non-trivial local branching (h, k) of X, there exist a unique critical branching (f, g) of X and a unique whisker $C[x]$ of X^* such that $(h, k) = (C[f], C[g])$.

A branching (f, g) of X is *confluent* if there exist n-cells f' and g' in X^* with the following shape

$$\begin{tikzcd}[row sep=small] & b \arrow[dr, bend left, "f'"] & \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & & a'. \\ & c \arrow[ur, bend right, "g'"'] & \end{tikzcd}$$

Note that trivial local branchings are always confluent. If a is an $(n-1)$-cell of X, we say that X is *confluent* (resp. *locally confluent*, resp. *critically confluent*) *at* a if every branching (resp. local branching, resp. critical branching) of source a is confluent. We say that X is *confluent* (resp.

locally confluent, resp. *critically confluent*) if it is at every $(n-1)$-cell of X^*. If X is confluent then every $(n-1)$-cell of X^* admits at most one normal form.

2.5.4. Theorem. *Fix $n > 0$, and let X be an n-polygraph.*

- ***(i)*** (Newman's lemma [169]) *If X terminates, then X is confluent if, and only if, it is locally confluent.*
- ***(ii)*** (The critical branchings theorem [170, 126, 112]) *X is locally confluent if, and only if, it is critically confluent.*

Proof. Apply Propositions 3.2.2 and 3.2.3 with $Y = n\,\mathrm{Sph}(X^\top)$. □

2.5.5. Convergence. We say that X is *convergent* if it is both terminating and confluent. In that case, every $(n-1)$-cell a of X^* has a unique normal form, denoted by $\widehat{a}$, and we have $\overline{a} = \overline{b}$ in $\overline{X}$ if, and only if, $\widehat{a} = \widehat{b}$ holds in X^*. Thus, the normal form defines a section of the canonical projection $X^*_{n-1} \twoheadrightarrow \overline{X}$, mapping an $(n-1)$-cell a of $\overline{X}$ to the unique normal form of its representative $(n-1)$-cells in X^*, also denoted by $\widehat{a}$.

As a consequence, if an n-category $\mathcal{C}$ admits a convergent presentation X, then $\mathcal{C}$ is isomorphic to the n-category formed by all the reduced n-cells of X, with k-composition of a and b given by $\widehat{a \star_k b}$.

2.5.6. Knuth-Bendix completion. Let X be a terminating n-polygraph, equipped with a termination order $\leqslant$. A *Knuth-Bendix completion of* X *(with respect to $\leqslant$)* is an n-polygraph Y, with the same cells as X up to dimension $n-1$, and with a filtration $Y_n = \cup_{p \geqslant 0} Y_n^p$ of its set of n-cells, such that the following properties is satisfied:

- **(i)** $Y_n^0 = X_n$,
- **(ii)** for every critical branching

$$b \xleftarrow{\;f\;} a \xrightarrow{\;g\;} c$$

of Y_n^p, there exist normal forms b' of b and c' of c in Y_n^p such that either
  - $b' = c'$,
  - $b' > c'$ and Y_n^{p+1} contains an n-cell $y : b' \to c'$,
  - $b' < c'$ and Y_n^{p+1} contains an n-cell $y : c' \to b'$.

By definition, a Knuth-Bendix completion of X is convergent, and Tietze-equivalent to X. Moreover, it induces a construction, the Knuth-Bendix completion procedure [126], that starts with X and progressively examines all the critical branchings, adding new n-cells when obstructions to critical confluence are found.

2.5.7. Métivier-Squier reduction. An n-polygraph X is *reduced* if, for every n-cell x of X, then $s(x)$ is a normal form of $X \setminus \{x\}$, and $t(x)$ is a normal form of X. Given a convergent n-polygraph X, the *Métivier-Squier reduction of* X is the n-polygraph obtained by the procedure that successively performs the following operations:

- **(i)** replace every generating n-cell $x : a \to b$ by $\widehat{x} : a \to \widehat{b}$,
- **(ii)** remove duplicate n-cells, if any,
- **(iii)** remove every n-cell whose source contains the source of another n-cell.

This process, due to Métiver for term rewriting [162] and Squier for string rewriting [192, Theorem 2.4], produces a reduced convergent n-polygraph that is Tietze-equivalent to X.

2.6.1. Generating polygraphs and the word problem for categories. Let $\mathcal{C}$ be a category. A 1-polygraph X *generates* $\mathcal{C}$ if there exists a functor $\pi : X^* \twoheadrightarrow \mathcal{C}$ that is the identity on 0-cells and surjective on 1-cells. We usually consider that the projection $\pi : X^* \twoheadrightarrow \mathcal{C}$ is implicitly specified for a given generating 1-polygraph X of $\mathcal{C}$ and, if u is a 1-cell of X^*, we just write $\overline{u}$ instead of $\pi(u)$. Every category admits itself as a generating polygraph. We say that $\mathcal{C}$ is *finitely generated* if it admits a finite generating 1-polygraph (in particular, the category must have finitely many 0-cells).

The *word problem for* $\mathcal{C}$ is the problem of finding a generating 1-polygraph X for $\mathcal{C}$ together with an algorithm that decides, for any two 1-cells u and v of X^*, whether or not $\overline{u} = \overline{v}$ holds in $\mathcal{C}$, i.e. whether or not the 1-cells u and v represent the same 1-cell of $\mathcal{C}$. The word problem is undecidable in general for a given category $\mathcal{C}$, even if it is finitely generated.

2.6.2. Proposition. *If a category admits a finite convergent presentation, then its word problem is decidable.*

Proof. Fix $\mathcal{C}$ a category and X a convergent presentation of $\mathcal{C}$. For all 1-cells u and v in X^*, we have $\overline{u} = \overline{v}$ if, and only if, $\widehat{u} = \widehat{v}$. Moreover, if X is finite, then one can effectively decide if a 1-cell of X^* is reduced or not, thus one can effectively compute the normal form of every 1-cell of X^*. □

2.6.3. Termination and confluence in dimension 2. Assume that X is a 2-polygraph. A useful example of potential termination order is the *left degree-wise lexicographic order* (or *deglex* for short) generated by a given order $\leqslant$ on the 1-cells of X. It is defined by

$$\begin{aligned} u < v \qquad & \text{if } \|u\| < \|v\|, \\ uxv < uyw \qquad & \text{if } \|v\| = \|w\| \text{ and } x < y \text{ in } X. \end{aligned}$$

The deglex order is total if, and only if, the original order on X is total.

By case analysis on the source of critical branchings of X, we can conclude that they must have one of the following two shapes

$$\begin{tikzcd} {} \arrow[r, "u"'] \arrow[rrr, bend left=50] & {} \arrow[r, "v" description] \arrow[r, bend right=60] \arrow[r, phantom, shift left=5ex, "\alpha\Uparrow"] \arrow[r, phantom, shift right=5ex, "\beta\Downarrow"] & {} \arrow[r, "w"'] & {} \end{tikzcd}
\qquad\qquad
\begin{tikzcd} {} \arrow[r, "u"'] \arrow[rr, bend left=50] & {} \arrow[r, "v" description] \arrow[rr, bend right=50] \arrow[r, phantom, shift left=5ex, "\alpha\Uparrow"] & {} \arrow[r, "w"] \arrow[phantom, shift right=5ex, "\Downarrow\beta"] & {} \end{tikzcd}$$

where α and β are 2-cells of X. In particular, if X is finite, then it has finitely many critical branchings. If X is reduced, then the first case cannot occur since, otherwise, the source of α would be reducible by β. Moreover, u, v and w are reduced non-identities 1-cells. Indeed, they are reduced since, otherwise, at least one of the sources of α and of β would be reducible by another 2-cell. If v was an identity, then the branching would be trivial. And, if u (resp. w) was an identity, then the source of β (resp. α) would be reducible by α (resp. β).

2.6.4. Termination by derivations in dimension 3. A general method to prove termination of a 3-polygraph X is to exhibit a derivation d of X_2^* into a natural system on X_2^* with values in $\mathcal{C}om$ of the form $N(F, G, M)$, with $\mathcal{V} = \mathcal{O}rd$, as described in 2.3.7, and to check that some sufficient conditions are met.

Let X be a 3-polygraph. Consider:

(i) Two 2-functors $F : X_2^* \to \mathcal{O}rd$ and $G : (X_2^*)^{co} \to \mathcal{O}rd$, such that the ordered sets $F(x)$ and $G(x)$ are non empty for every 1-cell x of X.

(ii) A commutative monoid M in $\mathcal{O}rd$, whose addition is strictly monotone in both arguments, and whose order is wellfounded.

(iii) A derivation $d : X_2^* \to N(F, G, M)$.

If, for every 3-cell $\alpha : f \Rrightarrow g$ of X, the three inequalities

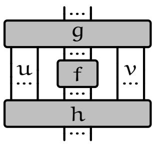
$$F(f) \geqslant F(g), \qquad G(f) \geqslant G(g), \qquad d(f) > d(g)$$

are satisfied, then X terminates [96, Theorem 4.2.1].

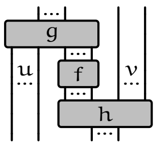
2.6.5. Confluence in dimension 3. This case is more difficult than in dimension 2, as initially noted in [135], mainly because a finite 3-polygraph may have an infinite number of critical branchings. However, an analysis of the possible shapes of these critical branching yields a sufficient condition for confluence that considers only a finite subset of them.

Assume that X is a 3-polygraph. By examination of the different possibilities, the critical branchings of X are classified as follows [96, §5.1.1]:

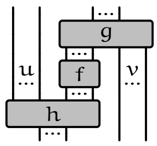
(i) *Inclusion* critical branchings (not possible if X is reduced), with the following source, if f is the source of a 3-cell of X, and $g \star_1 ufv \star_1 h$ is the source of another one:



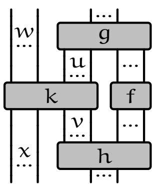
(ii) *Regular* critical branchings, with the following source, if $g \star_1 uf$ and $fv \star_1 h$, or $g \star_1 fv$ and $uf \star_1 h$ are the sources 3-cells of X:



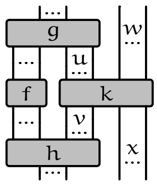
or



(iii) Instances of *left-indexed* critical branchings, with the following source, if $g \star_1 uf$ and $vf \star_1 h$ are the sources of 3-cells of X, and $k : wu \Rightarrow xv$ is a 2-cell of X^*:



(iv) Instances of *right-indexed* critical branchings, with the following source, if $g \star_1 fu$ and $fv \star_1 h$ are the sources of 3-cells of X, and $k : uw \Rrightarrow vx$ is a 2-cell of X^*:



(v) Instances of *multi-indexed* critical branchings, in all the other cases.

In the left-indexed or right-indexed case, the critical branching can be written $(C[k], D[k])$ for contexts $C[x]$ and $D[x]$ of X^*. The family $(C[k], D[k])_k$, where k ranges over the 2-cells with appropriate boundary, and such that $(C[k], D[k])$ is critical, is called a *right-indexed critical branching*. Each $(C[k], D[k])$ is an *instance* of the corresponding left-indexed or right-indexed critical branching, and this instance is *reduced* if k is reduced.

We say that X is *non-indexed* if it has inclusion or regular critical branchings only, *left-indexed* (resp. *right-indexed*) if it has inclusion, regular or left-indexed (resp. right-indexed) critical branchings only, and *finitely indexed* if each of its indexed critical branchings has a finite number of reduced instances.

Then we have the following results, used in §3.3.2 on a right-indexed 3-polygraph that presents the strict monoidal category of permutations:

(i) If X has a finite number of 3-cells, then it has a finite number of inclusion and regular critical branchings [96, Proposition 5.1.3].

(ii) If X is terminating and left-indexed (resp. right-indexed), then X is confluent if, and only if, all its inclusion and regular critical branchings, and all the reduced instances of its left-indexed (resp. right-indexed) critical branchings are confluent [96, Proposition 5.3.1].

2.6.6. Example. Let A be the two-element monoid $\{1, a\}$, with $a^2 = a$, seen as a 1-category with one 0-cell. The monoid A is presented by

$$\mathrm{As}_2 = (a_0 \mid a_1 : a_0 \to a_0 \mid a_2 : a_1 a_1 \Rightarrow a_1)\,.$$

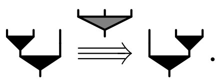
This 2-polygraph terminates, because $\|s(a_2)\| = 2 > 1 = \|t(a_2)\|$, and it has one critical branching, which is confluent:

$$\begin{tikzcd}[column sep=small]
& a_1 a_1 \arrow[dr, Rightarrow, "a_2"] & \\
a_1 a_1 a_1 \arrow[ur, Rightarrow, "a_2 a_1"] \arrow[dr, Rightarrow, "a_1 a_2"'] & & a_1 \\
& a_1 a_1 \arrow[ur, Rightarrow, "a_2"'] &
\end{tikzcd}$$

Thus As_2 is a convergent presentation of A.

Now, adjoin to As_2 a 3-cell, respectively given in classical notation and in string diagrams (with $a_2 =$ ) by

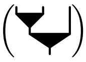
$$a_2 a_1 \star_1 a_2 \overset{a_3}{\Rrightarrow} a_1 a_2 \star_1 a_2$$

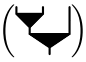


The resulting 3-polygraph As_3 is a convergent presentation of the theory of semigroups As, in the sense that the category of 2-functors from As to any monoidal category $\mathcal{C}$ is the same as the category of semigroups in $\mathcal{C}$. To prove that As_3 terminates, consider the natural system $\mathsf{N}(\mathsf{F}, *, \mathbb{N})$ with values in $\mathcal{C}om$ and the derivation d given by

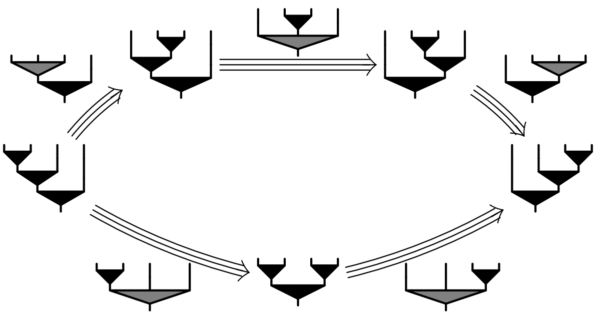
$$\mathsf{F}(|) = \mathbb{N} \setminus \{0\}, \qquad \mathsf{F}($$  $$)(i, j) = i + j, \qquad \mathsf{d}($$  $$)(i, j) = i.$$

To conclude, we check the required inequalities:

$$\mathsf{F}\Big($$  $$\Big)(i, j, k) = i + j + k = \mathsf{F}\Big($$  $$\Big)(i, j, k)$$

$$\mathsf{d}\Big($$  $$\Big)(i, j, k) = 2i + j > i + j = \mathsf{d}\Big($$  $$\Big)(i, j, k).$$

Finally, As_3 has one critical branching, and it is confluent, generating a 3-sphere with the shape of Stasheff's polytope K_4 [194]:



(2.2)

Chapter 3
Squier's theorem for higher categories

3.1. Introduction

3.1.1. Context. In the posthumous article [193], Squier proved two results (Theorems 5.2 and 5.3) which, rephrased in the language of higher categories and polygraphs from Chapter 2, respectively state: if a 2-polygraph X is a convergent presentation of a monoid M, then the confluence diagrams of the critical branchings of X form an acyclic extension of $X^{\top}$; as a consequence, if M admits a finite convergent presentation, then M is of finite derivation type. Since then, Squier's results and their proofs have been reworked several times, for example in [133, Theorem 2], or in [100, Theorems 4.3.2 and 4.3.3]. Over the time, the focus has shifted from Squier's Theorem 5.3, about the finiteness condition, to Theorem 5.2, giving an almost-constructive method to compute an acyclic extension. A more recent proof of Theorem 5.2 can be found in §4 of [95]; it is formulated in the case of associative algebras, but the arguments adapt to the case of monoids, and more generally to the case of n-categories thanks to the higher categorical setting of Chapter 2.

This chapter presents a higher-dimensional extension of Squier's results, from monoids, seen as 1-categories, to higher categories. This was motivated by concrete examples of strict monoidal categories, seen as 2-categories: Mac Lane's pros and props [153], Lawvere's theories [140] and algebraic operads [81, 144, 146], which are modern abstract descriptions of algebraic structures. Polygraphic presentations of such 2-categories were first studied by Burroni in his article introducing polygraphs [43], and their rewriting properties were first investigated by Lafont in [134, 135], followed by [88, 89]. For example, the structure of semigroup is encoded by the pro As just encountered in Example 2.6.6, and presented by the (convergent) 3-polygraph As_3. Now, consider the $(4,2)$-polygraph As_4 obtained from As_3 by adjunction of the 3-sphere (2.2), denoted by and corresponding to the unique critical branching of As_3: the $(3,2)$-category AsCat it presents is the theory of "associative categories", in the sense that 3-functors from AsCat to $\mathcal{C}at$, seen as a 3-category with one 0-cell and cartesian product as 0-composition, are the categories with a product that is associative up to a coherent natural isomorphism. Similarly to Mac Lane's coherence theorem for monoidal categories [152], a coherence theorem for associative categories would state that, in an associative category, all the diagrams formed by occurrences of the product and the associativity isomorphism commute; and a sufficient condition for this statement to be true would be for the 4-cell to form an acyclic extension of the free $(3,2)$-category $\mathrm{As}_3^{\top}$, so that AsCat would contain trivial 3-spheres only. Similar observations can be made to relate acyclic extensions of some $(3,2)$-categories to the coherence theorems for monoidal categories and symmetric monoidal categories [152]; or to relate partially acyclic extensions to the coherence theorem for braided monoidal categories, in which only some diagrams formed by the structural data commute [119].

Acyclic extensions are a homotopical formulation of the notion of syzygies between n-cells in higher categories. For example, in the case of presentations of groups, syzygies can be expressed as identities among relations. This notion originates in the work of Peiffer and Reidemeister in combinatorial group theory [175, 184], and is based on the crossed modules introduced by Whitehead in algebraic topology for the classification of homotopy 2-types [207, 208]. Crossed modules have also been defined for other algebraic structures than groups, such as commutative algebras [178], Lie algebras [123] or categories [177]. Then Baues has introduced track categories, now known as $(2,1)$-categories, as a model of homotopy 2-types [19, 18, 21], together with linear track extensions as a generalisations of crossed modules [22]. There exist several interpretations of identities among relations for presentations of groups: as homological 2-syzygies [40], as homotopical 2-syzygies [145] or as Igusa's pictures [145, 120]. One can also interpret identities among relations as the critical pairs of a group presentation by a convergent word rewriting system [58]; this point of view yields an algorithm based on Knuth-Bendix's completion procedure that computes a family of generators of the module of identities among relations [109].

3.1.2. Summary. First, §3.2 is devoted to the proof of the generalisation of Squier's theorem to higher categories. The proof given here is a modernised version of the one of §4.3 in [96]. It is adapted from the proof of Squier's theorem for associative algebras in [95], see Chapter 7, and based on the notion of Y-confluence: given an n-polygraph X and an extension Y of X^*, we say that a branching of X is Y-confluent if it is confluent, and the corresponding confluence diagram can be filled with an $(n+1)$-cell of $X^*(Y)$. So Y-confluence generalises confluence, in that confluence is the same as $n\,\mathrm{Sph}(X^\top)$-confluence, and both Newman's lemma and the critical branchings theorem can be extended accordingly. These two new results constitute the first two steps of the proof of Squier's theorem, the third and last one stating that Y-convergence implies acyclicity. To formulate the main result, we define a Squier completion of a convergent n-polygraph X as an $(n+1,n)$-polygraph $(X\,|\,Y)$ such that X is Y-convergent, and we obtain

Theorem 3.2.6. *Let X be a convergent presentation of an n-category $\mathcal{C}$. Then every Squier completion of X is a coherent presentation of $\mathcal{C}$.*

Then, §3.3 presents examples of applications of Squier's theorem, in dimension 2 and 3: a specific monoid, the monoidal category of permutations (using the study of critical branchings of 3-polygraphs of Chapter 2), and the standard coherent presentation of a category.

In §3.4, we give the definition of the finite derivation type homotopical property for higher categories, extending the case of monoids. We prove a technical result, Theorem 3.4.3, giving a way to transfer an acyclic extension from an n-polygraph X to an n-polygraph Y that presents the same $(n-1)$-category as X. From Theorem 3.4.3, we deduce that if two n-polygraphs present the same $(n-1)$-category, then both admit a finite acyclic extension, or neither does. The main result is

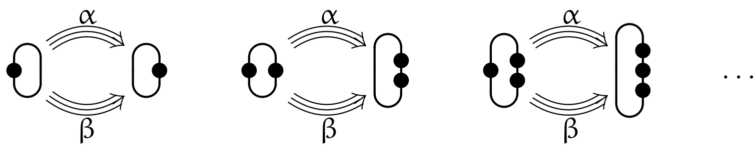
Theorem 3.4.6. ***(i)*** *For every $n > 0$, every finite convergent n-polygraph with a finite set of critical branchings if of finite derivation type. In particular, finite and convergent 2-*

polygraphs, and finite, convergent and finitely indexed 3-polygraphs are of finite derivation type.

(ii) *For every natural number $n \geqslant 3$, there exists a finite and convergent n-polygraph that is not of finite derivation type.*

Squier's theorem implies the first assertion. For the second point, we exhibited a counterexample, namely the 3-polygraph with one 0-cell, one 1-cell, three 2-cells , and , and four 3-cells

This 3-polygraph is finite and convergent, but all its acyclic extensions are infinite, due to an infinite number of reduced instances of indexed critical branchings:



As a consequence, the 2-category presented by this 3-polygraph is not of finite derivation type.

Next, §3.5 summarises applications of Squier's theorem to categorical coherence problems, described in [98]. We first introduce a higher version of Mac Lane's pros [153], and are mostly interested into $(3,2)$-pros, which are $(3,2)$-categories whose underlying category is the monoid $\mathbb{N}$ of natural numbers. Algebras over a $(3,2)$-pro $\mathcal{P}$ are defined as 3-functors from $\mathcal{P}$ into $\mathcal{C}at$, seen as a 3-category with only one 0-cell and cartesian product as 0-composition. In particular, there exists a $(3,2)$-pro As whose algebras are the monoidal categories, and similarly for symmetric monoidal categories and braided monoidal categories. A $\mathcal{P}$-diagram in a $\mathcal{P}$-algebra $\mathcal{A}$ is the image of a 3-sphere of $\mathcal{P}$ through $\mathcal{A}$. With this terminology, Mac Lane's coherence theorem for monoidal categories can be stated as: in all monoidal categories, all As-diagrams commute. The main result states that Squier's theorem gives a method to prove such a categorical coherence result:

Theorem 3.5.3. *Let X be a coherent presentation of a 2-pro, and let $\overline{X}$ be the $(3,2)$-pro presented by X. Then all $\overline{X}$-diagrams commute in every $\overline{X}$-algebra.*

We then prove that the usual definition of monoidal categories corresponds to a coherent presentation of the 2-pro of monoids, recovering Mac Lane's coherence result by a simple analysis of the critical branchings formed by associativity and the left and right neutrality laws, and then a reduction of the computed acyclic extension. In [98, §§3-4], this method is improved to obtain also the coherence theorems for symmetric and braided monoidal categories. It seems that Huet was the first to note the link between rewriting methods and coherence results for monoidal categories: in essence, §4 of [113] contains all the ideas explained here in §3.2 around the notion of coherent confluence and its relation to the coherence of categorical structures. Lucas further improved Theorem 3.5.3 to obtain a coherence theorem for pseudonatural transformations, whose structure is presented by a 3-polygraph that is not convergent [147].

Finally, §3.6 relates two different notions of syzygies for n-polygraphs: the homotopical ones, generated by an acyclic extension, and identities among relations. The latter are defined in [97] for an n-polygraph X as the elements of a certain natural system $\Pi(X)$ over the $(n-1)$-category presented by X. More precisely, $\Pi(X)$ is proved to be a quotient of the free natural system over the set of closed n-cells of $X^\top$ (i.e. the n-cells f such that $s(f) = t(f)$) by relations that force some compositions to be commutative. The main result of the section, Theorem 3.6.6, gives a way to compute generators of $\Pi(X)$ from an acyclic extension of an abelianised version $X^\top_{\mathrm{ab}}$ of $X^\top$, and conversely.

3.2. Coherent presentations from convergent presentations

3.2.1. Coherent confluence and convergence. Let X be an n-polygraph, and let Y be an extension of the $(n, n-1)$-category $X^\top$. A branching (f, g) of X is Y-*confluent* if there exist n-cells f' and g' in X^* and an $(n+1)$-cell F in $X^\top(Y)$ as in

$$\begin{tikzcd} & b \arrow[dr, bend left, "f'"] & \\ a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & \Downarrow F & a' \\ & c \arrow[ur, bend right, "g'"'] & \end{tikzcd}$$

The definitions of (local, critical) confluence and convergence are adapted to obtain *(local, critical)* Y-*confluence* and Y-*convergence*. The latter generalise the former, recovered for $Y = n\,\mathrm{Sph}(X^\top)$.

3.2.2. Proposition (The coherent Newman's lemma). *Let* X *be a terminating* n*-polygraph, and* Y *be an extension of* $X^\top$. *Then* X *is locally* Y*-confluent if, and only if, it is* Y*-confluent.*

Proof. Assume that X is locally Y-confluent. Let us prove that X is Y-confluent at every $(n-1)$-cell a of X^* by noetherian induction on a. If a is reduced, then $(1_a, 1_a)$ is the only branching of source a of X, and it is Y-confluent. Now, let a be a nonreduced $(n-1)$-cell of X^* such that X is Y-confluent at every 0-cell $b \prec a$, and let (f, g) be a branching of X of source a. If one of f or g is an identity, then (f, g) is Y-confluent. Otherwise, we construct

$$\begin{tikzcd} & b_1 \arrow[rr, bend left, "f_2"] \arrow[dr, "f_1'"] & & b \arrow[dr, bend left, "f_2'"] & \\ a \arrow[ur, bend left, "f_1"] \arrow[dr, bend right, "g_1"'] & \Downarrow F & a_1' \arrow[rr, "h"] & \Downarrow G & b' \arrow[dd, bend left, "k"] \\ & c_1 \arrow[ur, "g_1'"'] \arrow[dr, bend right, "g_2"'] & & \Downarrow H & \\ & & c \arrow[rr, bend right, "g_1'"'] & & a' \end{tikzcd}$$

Since f and g are not identities, they admit decompositions $f = f_1 \star_{n-1} f_2$ and $g = g_1 \star_{n-1} g_2$, where f_1 and g_1 are rewriting steps of X, and f_2 and g_2 are n-cells of X^*. We apply the local Y-confluence hypothesis on (f_1, g_1) to obtain f_1', g_1' and F, and then the induction hypothesis twice, first to (f_2, f_1') to obtain f_2', h and G, and then to $(g_1' \star_{n-1} h, g_2)$ to get k, g_1' and H. $\square$

3.2.3. Proposition (The coherent critical branchings theorem). *Assume that* X *is an* n*-polygraph, and that* Y *is an extension of* $X^\top$*. Then* X *is critically* Y*-confluent if, and only if, it is locally* Y*-confluent.*

Proof. First, we note that, in an n-polygraph, every trivial branching is $\emptyset$-confluent. Now, assume that X is critically Y-confluent, and prove that every non-trivial branching (f, g) is Y-confluent. Let (h, k) be the critical branching of X and C[x] the whisker of X* such that $(f, g) = (C[h], C[k])$. By hypothesis, (h, k) is Y-confluent: we apply C[x] to the resulting n-cells of X* and $(n+1)$-cell of $X^\top(Y)$ to get the result. $\square$

3.2.4. Proposition. *Let* X *be an* n*-polygraph, and* Y *be an extension of* $X^\top$*. If* X *is* Y*-convergent, then* Y *is acyclic.*

Proof. Since X is Y-convergent, it is convergent, so every $(n-1)$-cell a of X* admits a unique normal form $\widehat{a}$, and X* contains an n-cell $\eta_a : a \to \widehat{a}$.

Now, let $f : a \to b$ be an n-cell of X*. By hypothesis, the branching $(f \star_{n-1} \eta_b, \eta_a)$ is Y-confluent, so that, since $\widehat{a}$ and $\widehat{b}$ are reduced $(n-1)$-cells of X* that are necessarily equal, we get an $(n+1)$-cell

$$\begin{tikzcd} & b \arrow[dr, bend left, "\eta_b"] \arrow[d, phantom, "\Downarrow \eta_f"] & \\ a \arrow[ur, bend left, "f"] \arrow[rr, bend right, "\eta_a"'] & {} & \widehat{a} \end{tikzcd}$$

in $X^\top(Y)$. Put $\eta_{f^-} = f^- \star_{n-1} \eta_f^-$ to obtain an n-cell of $X^\top(Y)$ with the following shape:

$$\begin{tikzcd} & a \arrow[dr, bend left, "\eta_a"] \arrow[d, phantom, "\Downarrow \eta_{f^-}"] & \\ b \arrow[ur, bend left, "f^-"] \arrow[rr, bend right, "\eta_b"'] & {} & \widehat{a} \end{tikzcd}$$

Next, let $f : a \to b$ be an n-cell of $X^\top$. Write

$$f = g_1 \star_{n-1} h_1^- \star_{n-1} \cdots \star_{n-1} g_p \star_{n-1} h_p^-,$$

with each g_i and h_i in X*, and define η_f as the following composite n-cell of $X^\top(Y)$, with source $f \star_{n-1} \eta_b$ and target η_a:

$$\begin{tikzcd} a \arrow[r, "g_1"] \arrow[d, "\eta_a"'] \arrow[dr, phantom, "\swarrow \eta_{g_1}"] & b_1 \arrow[r, "h_1^-"] \arrow[d, "\eta_{b_1}"] \arrow[dr, phantom, "\swarrow \eta_{h_1^-}"] & a_2 \arrow[r] \arrow[d, "\eta_{a_2}"] & \cdots \arrow[r] & a_p \arrow[r, "g_p"] \arrow[d, "\eta_{a_p}"'] \arrow[dr, phantom, "\swarrow \eta_{g_p}"] & b_p \arrow[r, "h_p^-"] \arrow[d, "\eta_{b_p}"] \arrow[dr, phantom, "\swarrow \eta_{h_p^-}"] & b \arrow[d, "\eta_b"] \\ \widehat{a} \arrow[r, equal] & \widehat{a} \arrow[r, equal] & \widehat{a} \arrow[r, equal] & \cdots \arrow[r, equal] & \widehat{a} \arrow[r, equal] & \widehat{a} \arrow[r, equal] & \widehat{a} \end{tikzcd}$$

Finally, for all parallel 1-cells $f, g : a \to b$ of $X^\top$, the composite $(n+1)$-cell

$$\begin{tikzcd}
& b \arrow[dr, "\eta_b"] \arrow[drrr, bend left, "1_b"] \arrow[d, phantom, "\downarrow \eta_f"] & & & \\
a \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, "\eta_a"] & {} \arrow[d, phantom, "\downarrow \eta_g^-"] & \hat{a} \arrow[rr, "\eta_b^-"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & & b \\
& b \arrow[ur, "\eta_b"'] \arrow[urrr, bend right, "1_b"'] & & &
\end{tikzcd}$$

of $X^\top(Y)$ has source f and target g, thus concluding the proof that Y is acyclic. □

3.2.5. Squier completion. Let X be a convergent n-polygraph. A *Squier completion of* X is an $(n+1, n-1)$-polygraph Y, such that $Y_n = X$, and Y_{n+1} contains an $(n+1)$-cell

$$\begin{tikzcd}
& b \arrow[dr, "f'"] & \\
a \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\downarrow A"] & & a' \\
& c \arrow[ur, "g'"'] &
\end{tikzcd}$$

for every critical branching (f, g) of X, with f' and g' in X^*. Squier completions are not unique in general: the $(n+1)$-cell A could be directed in the reverse way, and the n-cells f' and g' chosen differently.

Composing Propositions 3.2.2, 3.2.3 and 3.2.4 gives the generalisation of Squier's result [193, Theorem 5.2], from convergent 2-polygraphs to convergent n-polygraphs:

3.2.6. Theorem (Squier's theorem). *Let* X *be a convergent presentation of an* n*-category* $\mathcal{C}$. *Then every Squier completion of* X *is a coherent presentation of* $\mathcal{C}$.

3.3. Examples of applications

3.3.1. Coherent presentations of monoids. Consider the monoid M presented by the 2-polygraph

$$X = \left(x, y \,\middle|\, xyx \overset{\alpha}{\Longrightarrow} yy\right).$$

Termination of X is proved with the deglex order generated by $x < y$. The 2-polygraph X has one, non confluent critical branching $(\alpha yx, xy\alpha)$. A Knuth-Bendix completion Y of X is obtained by adjunction of the dotted 2-cell β in the leftmost diagram, which in turn generates a new, confluent, critical branching, on the right-hand side:

$$\begin{tikzcd}
& yyyx \arrow[dd, Rightarrow, dotted, bend left, "\beta"] \\
xyxyx \arrow[ur, Rightarrow, "\alpha yx"] \arrow[dr, Rightarrow, "xy\alpha"'] \arrow[r, phantom, "\Downarrow A"] & {} \\
& xyyy
\end{tikzcd}
\qquad
\begin{tikzcd}
& xyyyyx \arrow[dr, Rightarrow, "xy\beta"] & \\
yyyxyx \arrow[ur, Rightarrow, "\beta yx"] \arrow[dr, Rightarrow, "yyy\alpha"'] \arrow[rr, phantom, "\Downarrow B"] & & xyxyyy \arrow[dl, Rightarrow, "\alpha yyy"] \\
& yyyyy &
\end{tikzcd}$$

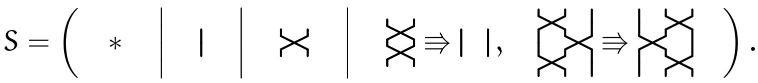
By Squier's theorem, the $(3,1)$-polygraph $Z = (x, y \mid \alpha, \beta \mid A, B)$ is a coherent presentation of M. Chapter 5 presents a reduction method to contract Z while preserving acyclicity: first, B can be shown to be superfluous, in the sense that $\{A\}$ is also an acyclic extension of $Y^{\top}$, and then β and A can be removed together to obtain that $(X \mid \emptyset)$ is a coherent presentation of M.

3.3.2. Coherent presentations of monoidal categories. Following the classification of critical branchings of 3-polygraphs given in §2.6.5, we obtain a smaller notion of Squier completion, improving Squier's theorem [96, Proposition 5.3.3]: assume that X is a convergent left-indexed (resp. right-indexed) 3-polygraph, and Y is an extension of $X^{\top}$ that contains one 4-cell

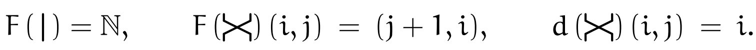
$$\begin{tikzcd}
 & b \arrow[dr, Rrightarrow, "F'"] & \\
a \arrow[ur, Rrightarrow, "F"] \arrow[dr, Rrightarrow, "G"'] \arrow[rr, phantom, "\Downarrow\omega"] & & a' \\
 & c \arrow[ur, Rrightarrow, "G'"'] &
\end{tikzcd}$$

for every pair (F, G) of 3-cells of X^* that is an inclusion or regular critical branching, or a reduced instance of a left-indexed (resp. right-indexed) critical branching, with F' and G' in X^*; then Y is acyclic.

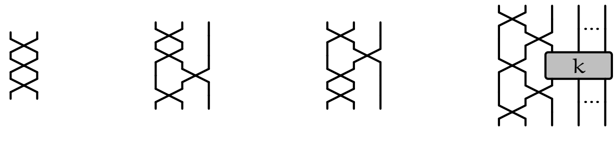
Consider, for example, the strict monoidal category $\mathbb{S}$ of permutations, seen as a 2-category with one 0-cell, the natural numbers as 1-cells, and one 2-cell $\tau : n \Rightarrow n$ for each permuation τ of $\{1, \ldots, n\}$. The 2-category $\mathbb{S}$ is presented by the 3-polygraph S, given in string diagrams as

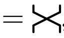


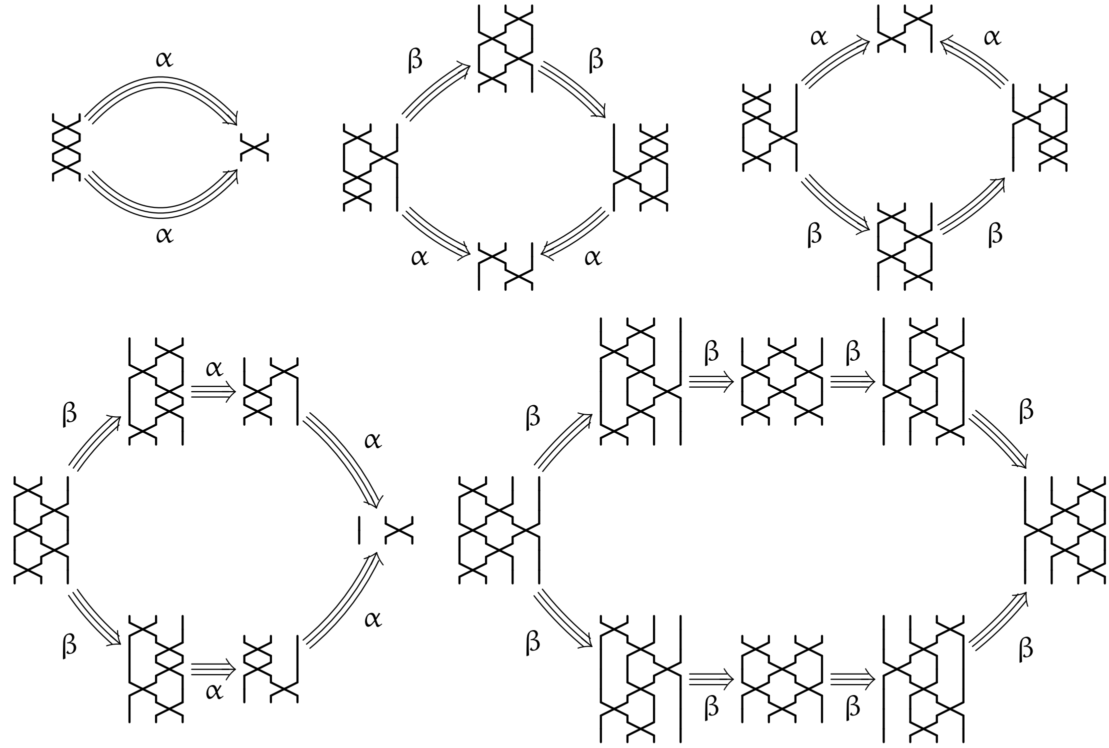
Termination of S is obtained by considering the natural system $N(F, *, \mathbb{N})$ on S_2^* with values in $\mathcal{C}om$, and the derivation d of S_2^* into $N(F, *, \mathbb{N})$ given by



The 3-polygraph S is right-indexed, with three regular critical branchings and one right-indexed critical branching, whose sources are



where k ranges over an infinite set of 2-cells. However, after classification of the normal forms of S, we obtain that the right-indexed branching has two reduced instances: for $k = |$ and $k =$ , see [96, §5.4]. The three regular critical branchings and the two reduced instances of the right-indexed critical branching are confluent, so S is convergent and we obtain a coherent presentation of $\mathcal{S}$ by filling the following five 3-spheres:



3.3.3. The standard coherent presentation. Fix a category $\mathcal{C}$. The *standard coherent presentation of* $\mathcal{C}$ is the $(3,1)$-polygraph $\mathrm{Std}^+(\mathcal{C})$ obtained by extension of the standard presentation of Example 2.4.5, with 3-cells

$$
\begin{tikzcd}
 & \widehat{ab}\,\widehat{c} \arrow[dr, Rightarrow, "\gamma_{ab,c}"] & \\
\widehat{a}\widehat{b}\widehat{c} \arrow[ur, Rightarrow, "\gamma_{a,b}\widehat{c}"] \arrow[dr, Rightarrow, "\widehat{a}\gamma_{b,c}"'] & \Downarrow \alpha_{a,b,c} & \widehat{abc} \\
 & \widehat{a}\,\widehat{bc} \arrow[ur, Rightarrow, "\gamma_{a,bc}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{1}_x\widehat{a} \arrow[dr, Rightarrow, "\gamma_{1_x,a}"] \arrow[d, phantom, "\Downarrow \lambda_a"] & \\
\widehat{a} \arrow[ur, Rightarrow, "\iota_x\widehat{a}"] \arrow[rr, equal] & {} & \widehat{a}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{a}\widehat{1}_y \arrow[dr, Rightarrow, "\gamma_{a,1_y}"] \arrow[d, phantom, "\Downarrow \rho_a"] & \\
\widehat{a} \arrow[ur, Rightarrow, "\widehat{a}\iota_y"] \arrow[rr, equal] & {} & \widehat{a}
\end{tikzcd}
$$

where $\alpha_{a,b,c}$ is indexed by the composable 1-cells a, b and c of $\mathcal{C}$, and λ_a and ρ_a, by the 1-cells $a : x \to y$ of $\mathcal{C}$. The $(3,1)$-polygraph $\mathrm{Std}^+(\mathcal{C})$ is, indeed, a coherent presentation of $\mathcal{C}$. It is not a Squier completion of $\mathrm{Std}(\mathcal{C})$, because the latter is not terminating: for every 0-cell x of $\mathcal{C}$, the 2-cell ι_x creates an infinite rewriting sequence $1_x \to \widehat{1}_x \to \widehat{1}_x\widehat{1}_x \cdots$. However, reversing all the 2-cells ι_x of $\mathrm{Std}(\mathcal{C})$ yields a convergent presentation of $\mathcal{C}$, and we obtain a Squier completion by adjoining all the 3-cells $\alpha_{a,b,c}$, and two families of 3-cells

$$
\begin{tikzcd}
\widehat{1}_x\widehat{a} \arrow[r, Rightarrow, bend left, "\gamma_{1_x,a}"] \arrow[r, Rightarrow, bend right, "\iota_x^-\widehat{a}"'] \arrow[r, phantom, "\Downarrow \lambda'_a"] & \widehat{a}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{a}\widehat{1}_y \arrow[r, Rightarrow, bend left, "\gamma_{a,1_y}"] \arrow[r, Rightarrow, bend right, "\widehat{a}\iota_y^-"'] \arrow[r, phantom, "\Downarrow \rho'_a"] & \widehat{z}
\end{tikzcd}
$$

indexed by the 1-cells $a : x \to y$ of $\mathcal{C}$. We deduce the result by reversing again the direction of the 2-cells ι_x^-, and, then, by replacing λ'_a and ρ'_a by $\lambda_a = \iota_x \widehat{a} \star_1 \lambda'_a$ and $\rho_a = \widehat{a}\iota_y \star_1 \rho'_a$.

3.4. Higher categories of finite derivation type

3.4.1. Finite derivation type. We say that an n-polygraph X is of *finite derivation type* if it is finite and if $X^\top$ admits a finite acyclic extension. An n-category $\mathcal{C}$ is of *finite derivation type* if it admits a finite coherent presentation.

Let us prove that finite derivation type is an invariant of finitely presented n-categories: all or none of the finite presentations of the same n-category are of finite derivation type. The main argument is Theorem 3.4.3 that describes how to transfer acyclic extensions from one n-polygraph to another one, for which we first require:

3.4.2. Lemma. *Let $\mathcal{C}$ be an $(n-1)$-free n-category, and let X and Y be presentations of $\mathcal{C}$. There exist $(n+1)$-functors*

$$F : X^\top \to Y^\top \qquad \textit{and} \qquad G : Y^\top \to X^\top$$

and, for all n-cells a of $X^\top$ and b of $Y^\top$, there exist $(n+1)$-cells

$$\sigma_a : GF(a) \to a \qquad \textit{and} \qquad \tau_b : FG(b) \to b$$

in $X^\top$ and $Y^\top$, respectively, such that the following conditions are satisfied:

(i) *the $(n+1)$-functors F and G induce the identity through the canonical projections onto $\mathcal{C}$:*

$$\begin{tikzcd} X^\top \arrow[r, two heads, "\pi_X"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & \mathcal{C} \arrow[d, "\mathrm{id}_\mathcal{C}"] \\ Y^\top \arrow[r, two heads, "\pi_Y"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X^\top \arrow[r, two heads, "\pi_X"] \arrow[dr, phantom, "="] & \mathcal{C} \\ Y^\top \arrow[r, two heads, "\pi_Y"'] \arrow[u, "G"] & \mathcal{C} \arrow[u, "\mathrm{id}_\mathcal{C}"'] \end{tikzcd}$$

(ii) *the $(n+1)$-cells σ_a and τ_b are functorial in a and b:*

$$\sigma_{1_a} = 1_{1_a}, \qquad \tau_{1_b} = 1_{1_b}, \qquad \sigma_{a \star_k a'} = \sigma_a \star_k \sigma_{a'} \quad \textit{and} \quad \tau_{b \star_k b'} = \tau_b \star_k \tau_{b'}.$$

Proof. First, define F, the case of G being symmetric. On a k-cell x of X, for $k < n$, put $F(x) = x$. Now, if $x : a \to b$ is an n-cell of X, we choose, in an arbitrary way, a representative n-cell $F(x) : a \to b$ of $\pi_X(x)$ in $Y^\top$, so that $\pi_Y(F(x)) = \pi_X(x)$ holds. Then, we extend F to every n-cell of $X^\top$ by functoriality. Let $x : a \to b$ be an $(n+1)$-cell of X. Since X is a presentation of $\mathcal{C}$, we have $\pi_X(a) = \pi_X(b)$, so that $\pi_Y(F(a)) = \pi_Y(F(b))$ holds. Using the fact that Y is a presentation of $\mathcal{C}$, we can choose an $(n+1)$-cell $F(x) : F(a) \to F(b)$ in $Y^\top$ and, then, extend F to every $(n+1)$-cell of $X^\top$ by functoriality.

Now, define σ, the case of τ being symmetric. Let x be an n-cell of X. By construction of F and G, we have $\pi_X(GF(x)) = \pi_Y F(x) = \pi_X(x)$. Since X is a presentation of $\mathcal{C}$, we can choose an $(n+1)$-cell $\sigma_x : GF(x) \to x$ in $X^\top$, and, then, extend σ to every n-cell of $X^\top$ as required. $\square$

3.4.3. Theorem (Transfer of acyclic extensions). *Let $\mathcal{C}$ be an $(n-1)$-free n-category, let X and Y be presentations of $\mathcal{C}$, and let F, G and τ be chosen as in Lemma 3.4.2. If Z is an acyclic extension of $X^\top$, then $F(Z) \sqcup \tau_Y$ is an acyclic extension of $Y^\top$, where:*

- ***(i)*** *the extension $F(Z)$ contains one $(n+2)$-cell $F(z) : F(a) \to F(a')$ for every $(n+2)$-cell $z : a \to a'$ of Z,*
- ***(ii)*** *the extension τ_Y contains one $(n+2)$-cell*

$$\begin{tikzcd}
 & FG(b') \arrow[dr, "\tau_{b'}"] & \\
FG(b) \arrow[ur, "FG(y)"] \arrow[dr, "\tau_b"'] & \Downarrow \tau_y & b' \\
 & b \arrow[ur, "y"'] &
\end{tikzcd}$$

for every $(n+1)$-cell $y : b \to b'$ of Y.

Proof. First, let us extend τ to every $(n+1)$-cell $g : b \to b'$ of $Y^\top$, to obtain an $(n+2)$-cell

$$\tau_g : FG(g) \star_n \tau_{b'} \to \tau_b \star_n g$$

of $Y^\top(\tau_Y)$. For that, we set $\tau_{1_b} = 1_{\tau_b}$ and $\tau_{g \star_k g'} = \tau_g \star_k \tau'_g$, for $k < n$. Then, for $g : b \to b'$ and $g' : b' \to b''$, we put

$$\tau_{b \star_n b'} \; = \; \begin{tikzcd}
 & & FG(b'') \arrow[dr, "\tau_{b''}"] & \\
 & FG(b') \arrow[ur, "FG(g')"] \arrow[dr, "\tau_{b'}"] & \Downarrow \tau_{g'} & b'' \\
FG(b) \arrow[ur, "FG(g)"] \arrow[dr, "\tau_b"'] & \Downarrow \tau_g & b' \arrow[ur, "g'"'] & \\
 & b \arrow[ur, "g"'] & &
\end{tikzcd}$$

Finally, for every $g : b \to b'$, we set

$$\tau_{g^-} \; = \; \begin{tikzcd}
 & & b \arrow[dr, "g"] & & \\
FG(b') \arrow[r, "FG(g)^-"] & FG(b) \arrow[ur, "\tau_b"] \arrow[dr, "FG(g)"'] & \Downarrow \tau_g^- & b' \arrow[r, "g^-"] & b \\
 & & FG(b') \arrow[ur, "\tau_{b'}"'] & &
\end{tikzcd}$$

There remains to check that τ_g is well defined, i.e. that it is compatible with the relations on $(n+1)$-cells in an $(n+2,n)$-category.

Second, consider an $(n+1)$-sphere $(g, g') : b \to b'$ of $Y^\top$. Because Z is an acyclic extension of $X^\top$, there exists an $(n+2)$-cell $A : G(g) \to G(g')$ in $X^\top(Z)$, yielding an

$(n+2)$-cell $F(A) : FG(g) \to FG(g')$ in $Y^{\top}(F(Z))$. As a consequence, the composite

$$\widetilde{A} \;=\; \begin{tikzcd} & b \arrow[rr, bend left, "g"] \arrow[d, phantom, "\downarrow\tau_g^-"] & & b' \\ FG(b) \arrow[ur, bend left, "\tau_b"] \arrow[r, bend left, "FG(g)"] \arrow[r, phantom, "\downarrow F(A)"] \arrow[r, bend right, "FG(g')"'] \arrow[dr, bend right, "\tau_b"'] & FG(b') \arrow[rr, "\tau_{b'}"] \arrow[d, phantom, "\downarrow\tau_{g'}"] & & b' \\ & b \arrow[urr, bend right, "b'"'] & & \end{tikzcd}$$

is a well-defined $(n+2)$-cell of $Y^{\top}(F(Z) \sqcup \tau_Y)$, and $\tau_b^- \star_n \widetilde{A}$ has (g, g') as boundary. $\square$

3.4.4. Corollary. *Let X and Y be Tietze-equivalent finite n-polygraphs, for $n > 0$. Then $X^{\top}$ admits a finite acyclic extension if, and only if, $Y^{\top}$ does.*

3.4.5. Remark. The finiteness condition is necessary in Corollary 3.4.4. For example, let X be the 2-polygraph with one 0-cell, three 1-cells a, b and c, and two 2-cells

$$ab \overset{\alpha}{\Longrightarrow} ba \qquad \text{and} \qquad ac \overset{\beta}{\Longrightarrow} a.$$

The polygraph X terminates (use the deglex order generated by $a > b$), and has no critical branching: Squier's theorem says that $(X\,|\,\emptyset)$ is a finite coherent presentation of the monoid $\overline{X}$ presented by X.

Now, consider the infinite 2-polygraph Y with the same cells as X up to dimension 1, and

$$ba \overset{\alpha'}{\Longrightarrow} ab \qquad \text{and} \qquad \left(ab^n c \overset{\beta_n}{\Longrightarrow} ab^n\right)_{n \geqslant 0}.$$

The 2-polygraph Y is Tietze-equivalent to X, because $\beta_0 = \beta$ and, for each $n > 0$, the 3-cell β_n relates 1-cells that are equal in $\overline{X}$: $ab^n c = b^n ac = b^n a = ab^n$. Termination of Y can be obtained with the deglex order generated by $b > a$, and Y has one critical branching, confluent, for each $n \geqslant 0$:

$$\begin{tikzcd} & ab^{n+1}c \arrow[dr, Rightarrow, "\beta_{n+1}"] \arrow[d, phantom, "\Downarrow A_n"] & \\ bab^n c \arrow[ur, Rightarrow, "\alpha'"] \arrow[dr, Rightarrow, "\beta_n"'] & {} & ab^{n+1} \\ & bab^n \arrow[ur, Rightarrow, "\alpha'"'] & \end{tikzcd}$$

Squier's theorem implies that $Z = \{A_n \,|\, n \in \mathbb{N}\}$ is an infinite acyclic extension of $Y^{\top}$. If $Y^{\top}$ admitted a finite acyclic extension, then there would exist a finite subset Z_0 of Z that would be acyclic [96, Proposition 3.2.3]. One checks that this is impossible, either by using a derivation, as in Example 4.3.10 of [96], or by observing that Y has no critical triple branching (see Chapter 4).

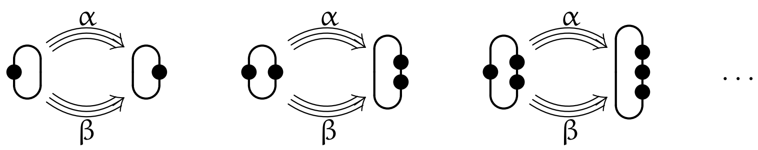
3.4.6. Theorem. ***(i)*** *For every $n > 0$, every finite convergent n-polygraph with a finite set of critical branchings is of finite derivation type. In particular, finite and convergent 2-polygraphs, and finite, convergent and finitely indexed 3-polygraphs are of finite derivation type.*

(ii) *For every natural number $n \geqslant 3$, there exists a finite and convergent n-polygraph that is not of finite derivation type.*

Proof. The first assertion is a direct consequence of Squier's theorem: if X is a finite convergent 2-polygraph, then its set of critical branchings is finite; the case of 3-polygraphs comes from the improved version of Squier's theorem [96, Proposition 5.3.3], see §3.3.2.

For the second assertion, we consider the 3-polygraph X with one 0-cell, one 1-cell, three 2-cells ϕ, $\cap$ and $\cup$, and four 3-cells

In §5.5 of [96], it is proved that X is finite and convergent, but not finitely indexed, and that $X^{\top}$ does not admit a finite acyclic extension, because each one must contain the 4-cells



To obtain the result in dimension $n \geqslant 4$, we lift the 3-polygraph X by adding two cells in each dimension up to $n - 4$. □

3.4.7. Squier's counterexample. In [192], Squier introduced a family $(S_k)_{k \geqslant 1}$ of monoids, where S_k is generated by $a, b, t, x_1, \ldots, x_k, y_1, \ldots, y_k$, submitted to the relations

$$at^n b = 1, \quad x_i a = atx_i, \quad x_i t = tx_i, \quad x_i b = bx_i, \quad x_i y_i = 1.$$

He proved, in [192] for $k \geqslant 2$, and in [193] for $k = 1$, that the finitely presented monoid S_k has a decidable word problem, but admits no finite convergent presentation. The proof of [192] uses homological arguments (see Chapter 4), while the proof of [193] relies on the property of finite derivation type. The same methods apply to the simpler example

$$\mathsf{LP} = \left\langle a, b, c, d, d' \,\middle|\, ab = a, da = ac, d'a = ac \right\rangle^{+}$$

given later in [139], with the same conclusions. The proof, translated in polygraphic terms in [100, §6.2], for $M = S_k$ or $M = \mathsf{LP}$, is composed of the following steps. First, use Knuth-Bendix's completion procedure on the finite presentation X of M to obtain an infinite convergent presentation Y of M. Then, apply Squier's theorem to obtain an infinite acyclic extension of $Y^{\top}$, and Theorem 3.4.3 to transfer it back to $X^{\top}$. Next, prove that the infinite acyclic extension so obtained cannot be reduced to a finite one (for example, using an invariant such as a derivation), to conclude that X is not of finite derivation type. Finally, use Corollary 3.4.4 to conclude that M is not of finite derivation type, and, as a consequence of Theorem 3.4.6, admits no finite convergent presentation.

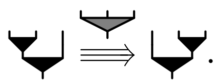
3.5.1. Higher pros and their algebras. For $0 < p \leqslant n \leqslant \infty$, an (n,p)-*pro* is an (n,p)-category $\mathcal{P}$ whose underlying 1-category is the monoid $\mathbb{N}$ of natural numbers with addition. In particular, $(2,2)$-pros, or 2-pros for short, coincide with Mac Lane's pros, see [153].

Fix a $(3,2)$-pro $\mathcal{P}$. A $\mathcal{P}$-*algebra* is a 3-functor from $\mathcal{P}$ to $\mathcal{C}at$, seen as 3-category with one 0-cell, categories as 1-cells, functors as 2-cells, natural transformations as 3-cells, and cartesian product as 0-composition. The $\mathcal{P}$-algebras and natural transformations between them form a category, denoted by $\mathcal{A}lg(\mathcal{P})$.

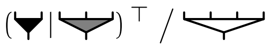
3.5.2. Coherence problems for algebras over a $(3,2)$-pro. Let $\mathcal{P}$ be a $(3,2)$-pro and let $\mathcal{A}$ be a $\mathcal{P}$-algebra. A $\mathcal{P}$-*diagram in* $\mathcal{A}$ is the image $\mathcal{A}(A,B)$ of a 3-sphere (A,B) of $\mathcal{P}$. A $\mathcal{P}$-diagram $\mathcal{A}(A,B)$ in $\mathcal{A}$ *commutes* if the relation $\mathcal{A}(A) = \mathcal{A}(B)$ is satisfied in $\mathcal{C}at$. Typical coherence theorems state that all, for every $\mathcal{P}$-algebra $\mathcal{A}$, all $\mathcal{P}$-diagrams (or a determined class of $\mathcal{P}$-diagrams) commute. Thus, the following result immediate:

3.5.3. Theorem. *Let* X *be a coherent presentation of a* 2*-pro, and let* $\overline{X}$ *be the* $(3,2)$*-pro presented by* X*. Then all* $\overline{X}$*-diagrams commute in every* $\overline{X}$*-algebra.*

3.5.4. Example. As in Example 2.6.6, consider the 2-pro As of semigroups, presented by the 3-polygraph As_2 with one 2-cell  and one 3-cell



As already seen, As is convergent, with exactly one critical branching, yielding by Squier's theorem an acyclic extension of $\mathrm{As}^\top$ with one 4-cell , filling Mac Lane's pentagon. The resulting $(4,2)$-polygraph being a presentation of the $(3,2)$-pro

AsCat = 

of associative categories (i.e. categories with a product that is associative up to a coherent natural isomorphism), we obtain that every AsCat-diagram commutes in every associative category.

3.5.5. Coherence for monoidal categories. A *monoidal category* is a category $\mathcal{C}$, equipped with two functors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $e : * \to \mathcal{C}$, and three natural isomorphisms

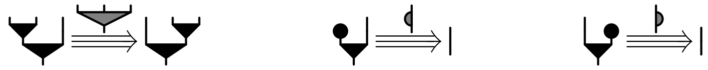
$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z), \qquad \lambda_x : e \otimes x \to x, \qquad \rho_x : x \otimes e \to x,$$

such that the following two diagrams commute in $\mathcal{C}$:

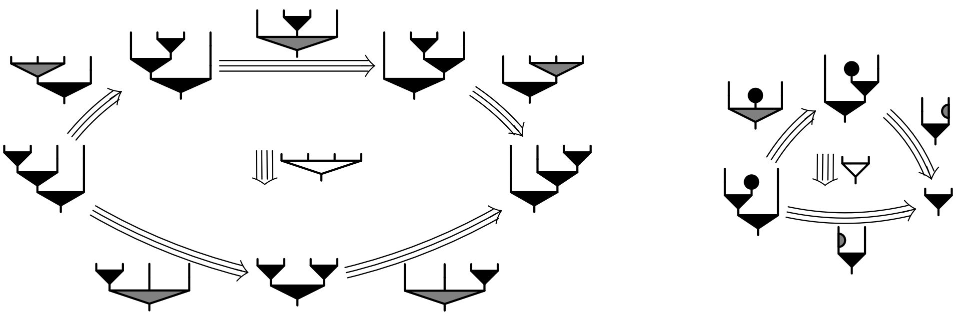
$$\begin{tikzcd}
& (x \otimes (y \otimes z)) \otimes t \arrow[r, "\alpha"] & x \otimes ((y \otimes z) \otimes t) \arrow[dr, "\alpha"] & \\
((x \otimes y) \otimes z) \otimes t \arrow[ur, "\alpha"] \arrow[drr, "\alpha"'] & & & x \otimes (y \otimes (z \otimes t)) \\
& & (x \otimes y) \otimes (z \otimes t) \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \otimes (e \otimes y) \arrow[dr, "\lambda"] & \\
(x \otimes e) \otimes y \arrow[ur, "\alpha"] \arrow[rr, "\rho"'] & & x \otimes y
\end{tikzcd}$$

Informally, Mac Lane's coherence theorem for monoidal categories says that, in every monoidal category, all the diagrams whose arrows are built from $\otimes$, e, α, λ and ρ commute.

Let MonCat be the $(3,2)$-pro presented by the $(4,2)$-polygraph MonCat with two 2-cells , , three 3-cells



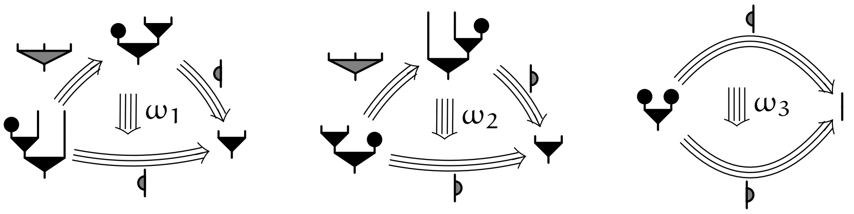
and two 4-cells:



It is straightforward to check that the categories $\mathcal{Alg}(\mathsf{MonCat})$ of MonCat-algebras and $\mathcal{MonCat}$ of monoidal categories are isomorphic. So, Mac Lane's coherence theorem says that, in every monoidal category, every MonCat-diagram commutes. This result is a direct consequence of:

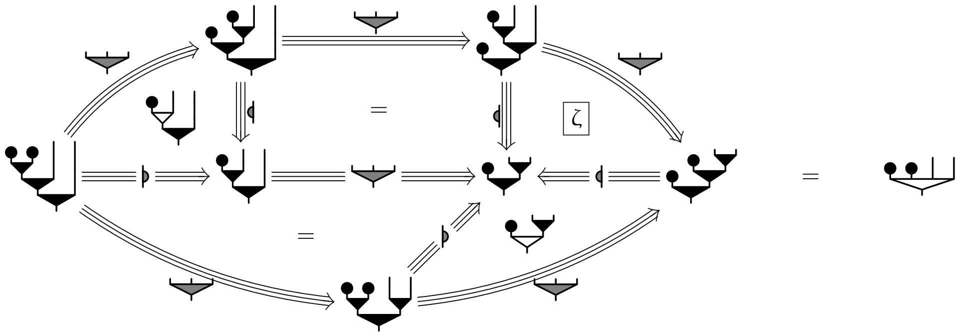
3.5.6. Proposition. *The* $(4,2)$*-polygraph* MonCat *is a coherent presentation of the* 2*-pro* Mon *of monoids.*

Proof. Denote by Mon the 3-polygraph underlying MonCat. The 2-pro Mon it presents is, indeed, the theory of monoids, in the sense that the 2-functors from Mon into any monoidal category $\mathcal{C}$ are exactly the monoids in $\mathcal{C}$. To prove that Mon terminates, we consider the natural system $N(F,*,\mathbb{N})$ on Mon* with values in $\mathcal{Com}$, and the derivation d of Mon* into $N(F,*,\mathbb{N})$ defined as in Example 2.6.6 on the 2-cell , extended with $F(\,)=1$ and $d(\,)=0$. The 3-polygraph Mon has five critical branchings. All of them are confluent, yielding, by Theorem 3.2.6, an acyclic extension of Mon$^\top$ with five 4-cells: the two 4-cells of MonCat, plus

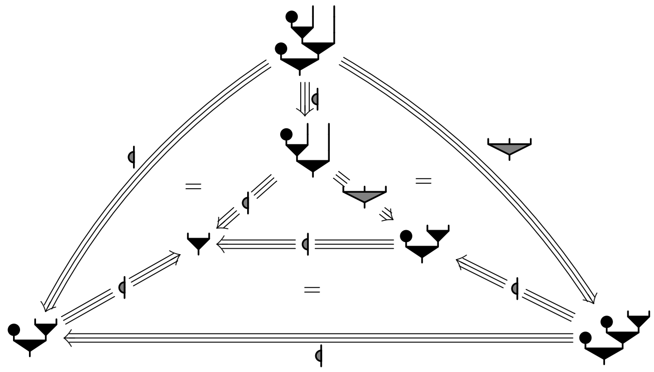


To prove that MonCat is a coherent presentation of Mon, there remain to show that, for each 4-cell ω_i, we have $\overline{s(\omega_i)} = \overline{t(\omega_i)}$ in $\mathcal{MonCat}$. For example, for ω_1, we define the 4-cell ζ of

MonCat$^{\top}$ by



where we abusively denote 3-cells by the generating 3-cell of Mon they contain. As a consequence of this construction, we have $\overline{s(\zeta)} = \overline{t(\zeta)}$ in MonCat. Then, we build the following diagram, relating the boundary of ω_1 (the inner triangle) to that of ζ (the outer triangle), thus proving that $\overline{s(\omega_1)} = \overline{t(\omega_1)}$ also holds:



See Proposition 2.3.3 in [98] for the cases of ω_2 and ω_3. □

3.5.7. Remark. The definition of monoidal category we have given is minimal, in the sense that both 4-cells of MonCat are required to get Mac Lane's coherence theorem. Indeed, assume that [pentagon 4-cell] is sufficient. Let d_1 be the derivation of Mon$^{\top}$ into the trivial module given by $d_1(◖) = 1$ and $d_1(▼) = d_1(◗) = 0$. Then $d_1(s(\text{pentagon})) = d_1(t(\text{pentagon})) = 0$. As a consequence, for every 4-cell Ω in Mon$^{\top}$(pentagon), we have $d_1(s(\Omega)) = d_1(t(\Omega))$. Thus, if {pentagon} was an acyclic extension of Mon$^{\top}$, we would have $d_1(A) = d_1(B)$ for every 3-sphere (A, B) of Mon$^{\top}$. But $d_1(s(\triangledown)) = 1$ and $d_1(t(\triangledown)) = 0$. Now, assume that $\triangledown$ is sufficient. Define the derivation d_2 of Mon$^{\top}$ into the trivial module by $d_2(▼) = 1$, $d_2(◖) = -1$, and $d_2(◗) = 0$. We have $d_2(s(\triangledown)) = d_2(t(\triangledown)) = 0$, but $d_2(s(\text{pentagon})) = 3$ and $d_2(t(\text{pentagon})) = 2$.

3.6. Coherent presentations and identities among relations

3.6.1. Abelian higher categories. In an n-category, a k-cell a is called *closed* if its source and target coincide, and, if $k \geqslant 2$ and $t(a)$ is an invertible $(k-1)$-cell, we define the closed $(k-1)$-cell $\widetilde{\partial}(a)$ by $\widetilde{\partial}(a) = s(a) \star_{n-1} t(a)^{-}$

Fix an $(n+1,n)$-category $\mathcal{C}$. If a is an n-cell of $\mathcal{C}$, we denote by $\mathrm{Aut}^{\mathcal{C}}_a$ the group of closed $(n+1)$-cells of $\mathcal{C}$ of source a. We say that $\mathcal{C}$ is *abelian* if, for every n-cell a of $\mathcal{C}$, the group $\mathrm{Aut}^{\mathcal{C}}_a$ is abelian. The *abelianised of* $\mathcal{C}$ the abelian $(n+1,n)$-category, denoted by $\mathcal{C}_{\mathrm{ab}}$, and defined as the quotient of $\mathcal{C}$ by the $(n+1)$-spheres $(f \star_n g, g \star_n f)$, where f and g are closed $(n+1)$-cells of $\mathcal{C}$ with the same source.

If $\mathcal{C}$ is abelian, the mapping $a \mapsto \mathrm{Aut}^{\mathcal{C}}_a$ extends to a natural system $\mathrm{Aut}^{\mathcal{C}}$ on the n-category $\mathcal{C}_n$, mapping a context $C[x]$ of $\mathcal{C}_n$ to the morphism of groups $y \mapsto C[y]$. This natural system satisfies the following properties:

3.6.2. Lemma ([97, Lemma 2.1.3]). *Let* $\mathcal{C}$ *be an* $(n+1,n)$*-category. For every* $(n+1)$*-cell* $g : b \to a$*, the application* $(\cdot)^g : \mathrm{Aut}^{\mathcal{C}}_a \to \mathrm{Aut}^{\mathcal{C}}_b$ *mapping* $f : a \to a$ *to*

$$f^g \;=\; g^- \star_{n-1} f \star_{n-1} g$$

is an isomorphism of groups. Moreover, if $\mathcal{C}$ *is abelian and* $g, h : b \to a$ *are* n*-cells of* $\mathcal{C}$*, then the isomorphisms* $(\cdot)^g$ *and* $(\cdot)^h$ *coincide.*

3.6.3. Identities among relations for polygraphs. Let X be an n-polygraph, for $n \geqslant 1$. Denote by $\Pi(X)$ the natural system on the $(n-1)$-category $\overline{X}$ presented by X given, on an $(n-1)$-cell a of $\overline{X}$ and up to isomorphism, by

$$\Pi(X)_a \simeq \mathrm{Aut}^{X^{\top}_{\mathrm{ab}}}_{\widehat{a}}, \tag{3.1}$$

where $\widehat{a}$ is any representative of a in X^*_{n-1}. The elements of the natural system $\Pi(X)$ are called the *identities among relations of* X.

Lemma 3.6.2 implies that the definition makes sense. Theorem 2.2.3 of [97] proves that $\Pi(X)$ is, up to isomorphism, the only natural system on $\overline{X}$ that satisfies (3.1), and that each $\Pi(X)_a$ admits the following presentation by generators and relations:

(i) one generator $\lfloor f \rfloor$ for every closed n-cell f of $X^{\top}$ such that $\overline{s(f)} = a$;

(ii) the relations are

$$\lfloor f \star_{n-1} g \rfloor = \begin{cases} \lfloor f \rfloor + \lfloor g \rfloor & \text{if } f \text{ and } g \text{ are closed,} \\ \lfloor g \star_{n-1} f \rfloor & \text{if } s(f) = t(g) \text{ and } t(f) = s(g). \end{cases}$$

Moreover, with this presentation, the isomorphism (3.1) is given by $\lfloor f \rfloor \mapsto f^g$, where $g : \widehat{a} \to s(f)$ is arbitrarily chosen in $X^{\top}$.

Lemma 3.4.2 and Theorem 3.4.3 are transposed to identities among relations in §2.3 of [97], giving

3.6.4. Proposition. *Assume that* X *and* Y *are Tietze-equivalent* n*-polygraphs, for* $n \geqslant 1$.

(i) *There exist* n*-functors* $F : X^{\top}_{ab} \to Y^{\top}_{ab}$ *and* $G : Y^{\top}_{ab} \to X^{\top}_{ab}$ *such that* $\pi_Y F = \pi_X$ *and* $\pi_X G = \pi_Y$*, and, for all* $(n-1)$*-cells* a *of* $X^{\top}_{ab}$ *and* b *of* $Y^{\top}_{ab}$*, there exist* n*-cells* $\sigma_a : GF(a) \to a$ *and* $\tau_b : FG(b) \to b$ *in* $X^{\top}_{ab}$ *and* $Y^{\top}_{ab}$ *that are functorial in* a *and* b.

(ii) *Assume that* F*,* G *and* τ *satisfy the previous conditions, and that* B *is a generating set for* $\Pi(X)$*. Then* $\lfloor F(B) \rfloor \sqcup \lfloor \tau_Y \rfloor$ *is a generating set of* $\Pi(Y)$*, where* $\lfloor F(B) \rfloor$ *contains one element* $\lfloor F(\beta) \rfloor$ *for every* $\beta \in B$*, and* $\lfloor \tau_Y \rfloor$ *contains one element*

$$\lfloor \tau_y \rfloor = \lfloor y \star_{n-1} \tau_b \star_{n-1} FG(y)^- \star_{n-1} \tau^-_{b'} \rfloor$$

for every n*-cell* $y : b \to b'$ *of* Y.

(iii) *If* X *and* Y *are finite, then* $\Pi(X)$ *is finitely generated if, and only if,* $\Pi(Y)$ *is finitely generated.*

Theorem 2.4.1 of [97] and Proposition 5.7.2 of [99] establish a correspondence between acyclic extensions of $X^{\top}_{\mathrm{ab}}$ and generators of $\Pi(X)$. Their proofs are based on the following technical result:

3.6.5. Lemma ([97, Lemma 1.3.6]). *Let* $\mathcal{C}$ *be an* $(n+1, n)$*-category, and* X *be an extension of* $\mathcal{C}$*. The following assertions are equivalent:*

(i) X *is acyclic.*

(ii) *Every closed* $(n+1)$*-cell* f *of* $\mathcal{C}$ *admits a decomposition*

$$f = (C_1[\widetilde{\partial}(x_1)^{\varepsilon_1}])^{g_1} \star_n \cdots \star_n (C_p[\widetilde{\partial}(x_p)^{\varepsilon_p}])^{g_p}$$

with $x_i \in X$*,* $\varepsilon_i \in \{-, +\}$*,* C_i *a whisker of* $\mathcal{C}$*, and* g_i *an* $(n+1)$*-cell of* $\mathcal{C}$*.*

3.6.6. Theorem. *Assume that* X *is an* n*-polygraph.*

(i) *If* Y *is an acyclic extension of* $X^{\top}_{ab}$*, then* $\Pi(X)$ *is generated by the set*

$$\left\lfloor \widetilde{\partial}(Y) \right\rfloor = \left\{ \left\lfloor \widetilde{\partial}(y) \right\rfloor \,\middle|\, y \in Y \right\}$$

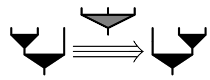
(ii) *If* B *is a generating set of* $\Pi(X)$*, then*

$$\left\{ \beta \to 1_{s(\beta)} \,\middle|\, \lfloor \beta \rfloor \in B \right\}$$

is an acyclic extension of $X^{\top}_{ab}$*.*

3.6.7. Example. Consider the 2-polygraph $\mathrm{As} = (a_0, a_1, a_2)$ of Example 2.6.6, and the acyclic extension of $\mathrm{As}^{\top}$ with one 3-cell, written in classical notation and in string diagrams, respectively,

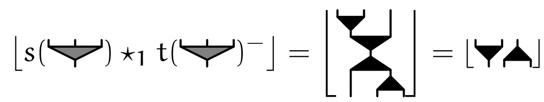
$$a_2 a_1 \star_1 a_2 \overset{a_3}{\Longrightarrow} a_1 a_2 \star_1 a_2$$



The abelian $(2,1)$-category $\mathrm{As}^{\top}_{\mathrm{ab}}$ is a quotient of $\mathrm{As}^{\top}$, and thus $\{a_3\}$ is also an acyclic extension of the former. Thus, by Theorem 3.6.6, the natural system $\Pi(X)$ is generated by one element:

$$\lfloor s(a_3) \star_1 t(a_3)^{-} \rfloor = \lfloor (a_2 a_1 \star_1 a_2) \star_1 (a_2^{-} \star_1 a_1 a_2^{-}) \rfloor = \lfloor a_2 a_1 \star_1 a_1 a_2^{-} \rfloor = \lfloor a_2 a_2^{-} \rfloor .$$

In string diagrams, writing  for $^{-}$, this generator is



3.6.8. Abelian finite derivation type. Fix $n \geqslant 1$, and an n-polygraph X. We say that X is of *abelian finite derivation type*, $\mathrm{FDT}_{\mathrm{ab}}$ for short, if it is finite and if the abelian $(n, n-1)$-category $X^{\top}_{\mathrm{ab}}$ admits a finite acyclic extension. By Theorem 3.6.6, if X is finite, then X is $\mathrm{FDT}_{\mathrm{ab}}$ if, and only if, $\Pi(X)$ is finitely generated. Since the later is invariant by Tietze-equivalence for finite n-polygraph by Proposition 3.6.4, the condition $\mathrm{FDT}_{\mathrm{ab}}$ is an invariant for finitely presented n-categories, and a necessary condition for being of finite derivation type. We conjecture that the converse implication is false.

Chapter 4
Squier's polygraphic resolution for categories

4.1. Introduction

4.1.1. Context. In [192] and, then, in [193], Squier has related the existence of a finite convergent presentation for a given monoid M to a homological finiteness condition, left-FP_3, and to a homotopical finiteness condition, finite derivation type, thereafter called FDT_3 for short. By his results, Squier has opened two different directions, one homological and one homotopical, to explore in the quest for a complete characterisation of the existence of finite convergent presentations of monoids. The corresponding invariants are related: FDT_3 implies left-FP_3, as proved by several authors [57, 181, 133]. The converse implication is false in general, as already noted by Squier in [193], yet it is true in the special case of groups [58], the latter result being based on the Brown-Huebschmann isomorphism between homotopical and homological syzygies [40]. However, the invariants left-FP_3 and FDT_3 are not complete characterisations of the property to admit a finite convergent presentation: they are necessary, but not sufficient conditions, as already proved by Squier in [193]. Following this observation, various refinements of both invariants have been explored.

In the homological direction, thanks to the notion of abelian resolution, one defines the more restrictive conditions left-FP_n, for every n in $\overline{\mathbb{N}}$: a monoid M has homological type left-FP_n if there exists a resolution of length n of the trivial left M-module by finitely generated and projective left M-modules. In [127], a notion of n-fold critical branching is used to complete the exact sequence used by Squier for the condition left-FP_3 into a resolution, obtaining the following implication: if a monoid admits a finite convergent presentation, then it is of homological type left-FP_n, for every n in $\overline{\mathbb{N}}$, the converse implication still being false in general, even for $n = \infty$. Similar results are also known for associative algebras [6], and for groups [55, 37, 87]. One can obtain other implications with the properties right-FP_∞ and bi-FP_∞, defined in terms of resolutions by right modules and bimodules, respectively.

In the homotopical direction, the condition FDT_3 has been refined into FDT_4 in [159], a property that can be rephrased as the existence of a finite presentation with a finite acyclic extension, itself satisfying a homotopical finiteness property. The condition FDT_4 is also necessary for a monoid to admit a finite convergent presentation and it is sufficient, but not necessary, for the conditions left/right/bi-FP_4. While the homological property FP_3 could be generalised in all dimensions using abelian resolutions, the property FDT_3 was not pushed further than FDT_4, possibly due to the lack of an appropriate notion of homotopical resolution.

In Chapter 3, we have formulated the condition FDT_3 as the existence of a finite coherent presentation, i.e. a 3-polygraph with a specific acyclicity condition. Coherent presentations have been generalised in all dimensions by Métayer in [160], leading to the concept of polygraphic resolutions of ∞-categories. In the special case of categories, the definition boils down to: an ∞-polygraph X is a polygraphic resolution of a category $\mathcal{C}$ if $\mathcal{C}$ is isomorphic to the category X_1^*/X_2 presented by X, and if, for every $n \geqslant 2$, the extension X_{n+1} of X_n^* is acyclic. Métayer has then proved that polygraphs are exactly the cofibrant objects in $\infty\mathcal{C}at$, for an adequate definition of cofibration [161], and, with Lafont and Worytkiewicz, that this homotopical point of view fits in a model structure on ∞-categories [138]: in this model structure, polygraphic resolutions are cofibrant approximations. Moreover, this model structure can be transferred easily to ∞-groupoids and, more generally, to (∞, p)-categories [9].

4.1.2. Summary. In §4.2, we define the $(\infty, 1)$-categorical version of the polygraphic resolutions of $\infty\mathcal{C}at$, and give a way to prove that a given $(\infty, 1)$-polygraph is a polygraphic resolution of a category. The latter is based on the polygraphic analogue of a contracting homotopy for a chain complex: this is a specific way to assign in a coherent way, to every n-cell x of an $(\infty, 1)$-polygraph X, an $(n+1)$-cell $\sigma_x : x \to \widehat{x}$, where $\widehat{x}$ is a kind of normal form for x. The description given in this chapter is a modernised version of the original one of [99, §3], introduced later in [95, §5] for polygraphic resolutions of associative algebras, and based on the treatment of Ara and Maltsiniotis in [8]. Contractions give a constructive way to characterise the acyclicity of an $(\infty, 1)$-polygraph:

Theorem 4.2.6. *Let X be an $(\infty, 1)$-polygraph. The canonical projection $X^\top \to \overline{X}$ is a weak equivalence in $(\infty, 1)\mathcal{C}at$ if, and only if, X admits a right ι-contraction for every unital section ι.*

Then, Squier's theorem for monoids is extended, in §4.3, to provide a construction of a polygraphic resolution $\mathrm{Sq}(X)$, called Squier's polygraphic resolution, from a convergent presentation X. If X is a reduced convergent 2-polygraph, the cells of $\mathrm{Sq}(X)$ correspond to the critical n-branchings of X, which are a generalised version of critical branchings where n generating 1-cells overlap. We obtain

Theorem 4.3.3. *Assume that X is a convergent presentation of a category $\mathcal{C}$. There exists a unique structure of $(\infty, 1)$-polygraph on $\mathrm{Sq}(X)$, and unique unital section ι and right ι-contraction σ of $\mathrm{Sq}(X)$, that satisfy $\iota_u = \widehat{u}$ for every 1-cell u of X^*, and, for every n-cell $u_1|\cdots|u_n$ of $\mathrm{Sq}(X)$, with $n \geqslant 1$, and every reduced 1-cell u_{n+1} of X^*,*

$$\sigma_{(u_1|\cdots|u_n)u_{n+1}} = \begin{cases} u_1|\cdots|u_{n+1} & \text{if } u_1|\cdots|u_{n+1} \in \mathrm{Sq}_{n+1}(X), \\ 1_{(u_1|\cdots|u_n)u_{n+1}} & \text{if } u_n u_{n+1} \text{ is reduced.} \end{cases}$$

As a consequence, with this structure of $(\infty, 1)$-polygraph, $\mathrm{Sq}(X)$ is a polygraphic resolution of $\mathcal{C}$.

Note that the original construction of [99, §4.5] only applied to reduced convergent presentations, and that the cells of the generalised version given here have a definition that is the same as the one used by Kobayashi in [127] to build abelian resolutions from convergent presentations. Moreover, the proof given here corrects an error in the one of Theorem 4.5.3 of [99]: the $(n+1)$-cell built at the end of §4.5.2 did not have the proper boundary to define a contraction. The notion of contraction and the construction of Squier's polygraphic resolution have, since then, been transposed to cubical (∞, p)-categories and their associated polygraphs by Lucas in [150, 149, 148]: the definitions and constructions are much clearer in this setting, due to the natural cubical shapes of the cells associated to the n-branchings, and to cubical symmetries generated by natural operations on the n-branchings.

Next, §4.4 starts with the construction, for a category $\mathcal{C}$ and an $(\infty, 1)$-polygraph X that presents $\mathcal{C}$, of an augmented chain complex $F_{\mathcal{C}}[X]$ of free natural systems over $\mathcal{C}$: this complex is formed of the free natural systems $F_{\mathcal{C}}[X_n]$ over each set X_n of n-cells of X, and its boundary map is given by the source and target maps: $\delta[x] = [s(x)] - [t(x)]$, where $[\cdot]$ is a derivation with respect to the 0-composition, and maps every other composition to the sum. This complex is inspired by constructions of Reidemeister [184], Fox [75] and Squier [192], and the *Fox Jacobian* used by Squier is the boundary map δ_2 of $F_{\mathcal{C}}[X]$. We obtain

Theorem 4.4.3. *If X is a polygraphic resolution of a category $\mathcal{C}$, then $F_{\mathcal{C}}[X]$ is a free resolution of the constant natural system $\mathbb{Z}$ on $\mathcal{C}$.*

The key argument of the proof is that, if X is a polygraphic resolution, then, by Theorem 4.2.6, it admits a contraction that, in turn, induces a contracting homotopy for $F_{\mathcal{C}}[X]$. Note that the construction of $F_{\mathcal{C}}[X]$ adapts in a straightforward way to left/right/bimodules over $\mathcal{C}$, with similar conclusions. In analogy with the case of groups, we define the homological n-syzygies of X as the kernel of the boundary map δ_n, and we establish

Theorem 4.4.6. *For every 2-polygraph X, the natural systems of homological 2-syzygies and of identities among relations of X are isomorphic.*

This isomorphism was inspired by Loday's result on the correspondence between four different definitions of syzygies for presentations of groups: homotopical 2-syzygies, Igusa's pictures, identities among relations, and homological 2-syzygies [145].

In §4.5, we use polygraphic resolutions to generalise Squier's and Pride's homotopical finiteness conditions FDT_3 and FDT_4 to all dimensions: a category is FDT_n if it admits a polygraphic resolution that contains finitely many cells in every dimension up to n. So, Theorem 4.3.3 immediately implies that a category with a finite convergent presentation is FDT_n for every n in $\overline{\mathbb{N}}$, because a finite convergent 2-polygraph has finitely many critical n-branchings. We also introduce a new homological finiteness condition, FP_n, based on the existence of a resolution of length n by finitely generated natural systems, which is a refinement of the analogous conditions defined in terms of left/right/bimodules. The known relations between all these finiteness conditions are summarised in the following two results:

Theorem 4.5.6. *Let $\mathcal{C}$ be a category. For every $n \in \overline{\mathbb{N}}$, if $\mathcal{C}$ is FDT_n, then it is FP_n. In particular, if $\mathcal{C}$ admits a convergent presentation, then it is FP_∞.*

Theorem 4.5.7. *Let $\mathcal{C}$ be a category, and X be a finite presentation of $\mathcal{C}$. The following assertions are equivalent: $\mathcal{C}$ is FP_3, $\mathcal{C}$ is $\mathrm{FDT}_{\mathrm{ab}}$, $h_2(X)$ is finitely generated, and $\Pi(X)$ is finitely generated.*

Finally, Section 4.6 applies the constructions defined so far on several examples, in particular: any category $\mathcal{C}$ with its reduced standard presentation, yielding the reduced standard polygraphic resolution of $\mathcal{C}$ which, after abelianisation, boils down to the classical bar resolution of $\mathcal{C}$; the monoid $A = \{1, a\}$ with product $a^2 = a$, admitting a polygraphic resolution made of the Stasheff polytopes; the category of monotone surjections on finite sets.

4.2. Polygraphic resolutions and contractions

4.2.1. Polygraphic resolutions in $(\infty, 1)\mathcal{C}at$. In an ∞-category $\mathcal{C}$, the *ω-equivalence relation*, denoted by $\sim_\omega$, is defined by coinduction on the dimension: two n-cells a and b of $\mathcal{C}$ are *ω-equivalent* if there exist $(n+1)$-cells $f : a \to b$ and $g : b \to a$ such that $f \star_n g \sim_\omega 1_a$ and $g \star_n f \sim_\omega 1_b$. In the *canonical* (or *folk*) model structure on $\infty\mathcal{C}at$, defined in [138], the weak equivalences are the ∞-functors $F : \mathcal{C} \to \mathcal{D}$ satisfying

(i) for every 0-cell x of $\mathcal{D}$, there exists a 0-cell $\widehat{x}$ of $\mathcal{C}$ such that $F(\widehat{x}) \sim_\omega x$,

(ii) for every n-sphere (a, b) of $\mathcal{C}$, and every $(n+1)$-cell $f : F(a) \to F(b)$ in $\mathcal{D}$, there exists an $(n+1)$-cell $\widehat{f} : a \to b$ in $\mathcal{C}$ such that $F(\widehat{f}) \sim_\omega f$.

The cofibrations of $\infty\mathcal{C}at$ are the retracts of transfinite compositions of pushouts of the canonical inclusions $\mathcal{S}^{n-1} \rightarrowtail \mathcal{E}^n$, and the cofibrant objects of $\infty\mathcal{C}at$ are exactly the ∞-polygraphs [161].

The canonical model structure of $\infty\mathcal{C}at$ transfers to $(\infty, 1)\mathcal{C}at$ through the adjunction formed by the enveloping $(\infty, 1)$-category functor and the forgetful functor. The proof is given in [9] for ∞-groupoids, but works equally well for (∞, p)-categories, for every $p \geqslant 0$, yielding in particular a model structure on $(\infty, 1)\mathcal{C}at$. By construction, the weak equivalences in $(\infty, 1)\mathcal{C}at$ are the images through the forgetful functor of the weak equivalences in $\infty\mathcal{C}at$, i.e. the ∞-functors between $(\infty, 1)$-categories that are weak equivalences in $\infty\mathcal{C}at$. The cofibrations in $(\infty, 1)\mathcal{C}at$ are the retracts of transfinite compositions of pushouts of the inclusions $(\mathcal{S}^{n-1})^\top \rightarrowtail (\mathcal{E}^n)^\top$. Thus, the free $(\infty, 1)$-category $X^\top$ over an (∞, p)-polygraph is a cofibrant object in $(\infty, 1)\mathcal{C}at$, see [99, Proposition 2.2.4].

Let $\mathcal{C}$ be a category. A *polygraphic resolution of $\mathcal{C}$ (in $(\infty, 1)\mathcal{C}at$)* is an $(\infty, 1)$-polygraph X such that $X^\top$ is a cofibrant approximation of $\mathcal{C}$ in $(\infty, 1)\mathcal{C}at$. Expanding the definition, X is a polygraphic resolution of $\mathcal{C}$ if, and only if, it presents $\mathcal{C}$ and, for every $n \geqslant 2$, the extension X_{n+1} of $X_n^\top$ is acyclic.

4.2.2. Homotopies. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be ∞-functors. A *homotopy from F to G* is a graded map

$$\mathcal{C} \xrightarrow{\eta} \mathcal{D}$$

of degree 1 that satisfies, writing η_a for $\eta(a)$,

(i) for every 0-cell a of $\mathcal{C}$,

$$s(\eta_a) = F(a) \qquad \text{and} \qquad t(\eta_a) = G(a),$$

(ii) for every $n \geqslant 1$, and every n-cell a of $\mathcal{C}$,

$$s(\eta_a) = F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(a)},$$
$$t(\eta_a) = \eta_{s_{n-1}(a)} \star_{n-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(a),$$

(iii) for all $n > k \geqslant 0$, and every k-composable pair (a, b) of n-cells of $\mathcal{C}$,

$$\begin{aligned}\eta_{a \star_k b} &= F(s_{k+1}(a)) \star_0 \eta_{t_0(b)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(b)} \star_k \eta_b \\ &\qquad \star_{k+1} \eta_a \star_k \eta_{s_{k-1}(a)} \star_{k-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(t_{k+1}(b)),\end{aligned}$$

(iv) for every $n \geqslant 0$, and every n-cell a of $\mathcal{C}$,

$$\eta_{1_a} = 1_{\eta_a}.$$

Several results must be established to ensure that this definition makes sense, and in particular that all the involved compositions are well defined, see [8, §B.8].

4.2.3. Unital sections. Let X be an $(\infty, 1)$-polygraph. For c a 1-cell of the quotient category $\overline{X}$, we denote by $X_c^\top$ the corresponding fibre of the canonical projection $\pi : X^\top \twoheadrightarrow \overline{X}$. By definition, $X_c^\top$ is an ∞-groupoid, whose 0-cells are the representatives of c in $X^\top$. To avoid confusion, we keep the dimensions of the $(\infty, 1)$-category $X^\top$ when talking about the cells and compositions of $X_c^\top$.

A *unital section of* X is a family

$$\left(* \xrightarrow{\iota} X_c^\top \right)_{c \in \overline{X}_1}$$

of ∞-functors, indexed by the 1-cells c of $\overline{X}$, that satisfy $\iota_{1_x} = 1_{1_x}$ for every 0-cell x of X. Concretely, ι assigns to every 1-cell c of $\overline{X}$ a representative 1-cell ι_c in $X^\top$, in such a way that identities are mapped to identities. A unital section of X is almost a functorial section of the canonical projection $\pi : X^\top \to \overline{X}$, but undefined in dimension 0 and with no specific compatibility with the 0-composition.

Fix a unital section ι of X. If a is an n-cell of $X^\top$, we write $\widehat{a}$ for $\iota\pi(a)$ when no confusion occurs (note that $\widehat{a}$ is an identity if $n \geqslant 2$). A 1-cell u of $X^\top$ is ι-*reduced* if $u = \widehat{u}$ holds. A non-ι-reduced 1-cell u of $X^\top$ is ι-*essential* if $u = xv$, with x a 0-cell of X and v an ι-reduced 1-cell of $X^\top$.

4.2.4. Contractions. Let X be an $(\infty, 1)$-polygraph, and ι be a unital section of X. An ι-*contraction of* X is a family

$$\left(\begin{tikzcd} X_c^\top \arrow[rr, bend left, "\mathrm{id}_{X_c^\top}"] \arrow[dr, "\pi"'] \arrow[rr, phantom, "\Downarrow \sigma"] & & X_c^\top \\ & * \arrow[ur, "\iota"'] & \end{tikzcd} \right)_{x \in X_c^\top}$$

of homotopies, indexed by the 1-cells c of $\overline{X}$, that satisfies $\sigma_a = 1_a$ for every n-cell a of $X^\top$, with $n \geqslant 1$, such that a belongs to the image of ι or of σ. Thus, an ι-contraction is almost a homotopy from $\mathrm{id}_{X^\top}$ to $\iota\pi$, but, like $\iota\pi$, undefined on 0-cells and with no specific compatibility with the 0-composition.

Fix an ι-contraction σ of X. The definition of σ implies, for every $n \geqslant 1$, every n-cell a of $X^\top$, and every $1 \leqslant k < n$,

$$s_k(\sigma_a) = a \star_1 \sigma_{t_1(a)} \star_2 \cdots \star_k \sigma_{t_k(a)} \qquad \text{and} \qquad t_k(\sigma_a) = \begin{cases} \widehat{a} & \text{if } k = 1, \\ \sigma_{s_k(a)} & \text{otherwise.} \end{cases}$$

We say that σ is *right* if, for every $n \geqslant 1$ and all n-cells f and g of $X^\top$ of respective 1-sources a and b, it satisfies

$$\sigma_{fg} = a\sigma_g \star_1 \sigma_{f\widehat{b}}. \tag{4.1}$$

The notion of left ι-contraction is defined symmetrically, and the following definitions and results admit a corresponding symmetric version.

For $n \geqslant 1$, an n-cell a of $X^\top$ is σ-*reduced* if it is an identity or in the image of σ. If σ is a right ι-contraction of X, and $n \geqslant 1$, a non-σ-reduced n-cell a of $X^\top$ is σ-*essential* if there exist an n-cell α of X and an ι-reduced 1-cell u of $X^\top$ such that $a = \alpha u$.

4.2.5. Lemma ([99, Corollary 3.3.5]). *Let X be an ∞-polygraph, and ι be a unital section of X. A right ι-contraction σ of X is uniquely and entirely determined by its values on the ι-essential 1-cells and, for every $n \geqslant 1$, on the σ-essential n-cells of $X^\top$.*

Proof. If σ is a right ι-contraction, then its values are prescribed on every cell of $X^\top$ that is not ι-essential or σ-essential. Now, the values of σ on ι-essential and σ-essential cells of $X^\top$ can be chosen freely (with correct source and target), provided that these values make σ compatible with all the defining relations of the structure of $(\infty, 1)$-category, and in particular with exchange relations between the 0-composition and the other compositions. It turns out that (4.1) imposes compatibility with these exchange relations. □

4.2.6. Theorem. *Let X be an $(\infty, 1)$-polygraph. The canonical projection $X^\top \to \overline{X}$ is a weak equivalence in $(\infty, 1)\mathcal{C}at$ if, and only if, X admits a right ι-contraction for every unital section ι.*

Proof. Assume that X is a polygraphic resolution of $\overline{X}$. Let us define a right ι-contraction σ of X thanks to Lemma 4.2.5. If xu is an ι-essential 1-cell of $X^\top$, then xu and $\widehat{xu}$ have the same image in $\overline{X}$, so that, by definition of $\overline{X}$, there exists a 1-cell $\sigma_{xu} : xu \to \widehat{xu}$ in $X^\top$. Assume that σ is defined on the n-cells of $X^\top$, for $n \geqslant 1$, and let αu be a σ-essential $(n+1)$-cell of $X^\top$. The n-cells $s(\sigma_{\alpha u})$ and $t(\sigma_{\alpha u})$ are parallel, so, by hypothesis, there exists an $(n+1)$-cell $\sigma_{\alpha u}$ with these source and target in $X^\top$.

Conversely, let σ be an ι-contraction of X, and a and b be parallel n-cells of $X^\top$, for $n \geqslant 1$. We have $t(\sigma_a) = \sigma_{s(a)} = \sigma_{s(b)} = t(\sigma_b)$ by hypothesis, so that the $(n+1)$-cell $\sigma_a \star_n \sigma_b^-$ is well defined, with source $s(\sigma_a)$ and target $s(\sigma_b)$. The fact that $t_k(a) = t_k(b)$ holds for every $0 \leqslant k < n$ implies that

$$(\sigma_a \star_n \sigma_b)^- \star_{n-1} \sigma^-_{t_{n-1}(a)} \star_{n-2} \cdots \star_0 \sigma^-_{t_0(a)}$$

is a well-defined $(n+1)$-cell of $X^\top$, with source a and target b, thus proving that X_{n+1} is acyclic. □

4.3. Polygraphic resolutions from convergent presentations

4.3.1. The cells of Squier's polygraphic resolution. Assume that X is a 2-polygraph. Define $\mathrm{Sq}(X)$ as the graded set whose 0-cells and 1-cells are the ones of X, and whose n-cells, for $n \geqslant 2$, are the families $(u_1, \ldots, u_n)$, written $u_1|\cdots|u_n$, of non-identity reduced 1-cells of X^* that satisfy

(i) u_1 is a 1-cell of X,
(ii) for every $1 \leqslant i < n$, the 1-cell $u_i u_{i+1}$ is not reduced,
(iii) for every $1 \leqslant i < n$, every proper left-factor of $u_i u_{i+1}$ is reduced.

4.3.2. Interpretation in the reduced case. Assume that X is a reduced 2-polygraph. Then $u_1|u_2$ is a 2-cell of $\mathrm{Sq}(X)$ if, and only if, u_1 belongs to X and $u_1 u_2$ is the source of a 2-cell of X.

From the analysis conducted in §2.6.3, the critical branchings of X are of the form

$$\begin{tikzcd}
{} \arrow[r, "u_1"'] \arrow[rrr, bend left=60] \arrow[rrr, phantom, bend left=35, "\alpha\Uparrow"] & {} \arrow[r, "v_2"'] & {} \arrow[r, "w_2"] \arrow[rr, bend right=60] \arrow[rr, phantom, bend right=35, "\Downarrow\beta"] & {} \arrow[r, "u_3"] & {}
\end{tikzcd}$$

where α and β are 2-cells of X, and $u_1 v_2$, w_2 and u_3 are reduced non-identity 1-cells of X, with u_1 in X. Putting $u_2 = v_2 w_2$ induces a bijection between the 3-cells $u_1|u_2|u_3$ of $\mathrm{Sq}(X)$ and the critical branchings of X whose source is $u_1 u_2 u_3$.

For $n \geqslant 3$, define the *critical* n*-branchings of* X as the non-ordered families $(a_1, \ldots, a_n)$ of rewriting steps of X with the same source, overlapping in a nontrivial and minimal way. Conducting a similar analysis as in §2.6.3 shows that the critical 3-branchings of X fall in one of the two cases

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "u_1"'] \arrow[rrrr, bend left=50] \arrow[rrrr, phantom, bend left=30, "\alpha\Uparrow"] & {} \arrow[r, "v_2"'] & {} \arrow[r, "w_2"] \arrow[rrr, bend right=50] \arrow[rrr, phantom, bend right=30, "\Downarrow\beta"] & {} \arrow[r, "x_2"] \arrow[rrr, bend left=50] \arrow[rrr, phantom, bend left=30, "\Uparrow\gamma"] & {} \arrow[r, "u_3"] & {} \arrow[r, "u_4"'] & {}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}[column sep=small]
{} \arrow[r, "u_1"'] \arrow[rrr, bend left=60] \arrow[rrr, phantom, bend left=35, "\alpha\Uparrow"] & {} \arrow[r, "v_2"'] & {} \arrow[r, "w_2"] \arrow[rrr, bend right=50] \arrow[rrr, phantom, bend right=30, "\Downarrow\beta"] & {} \arrow[r, "v_3"] & {} \arrow[r, "w_3"] \arrow[rr, bend left=60] \arrow[rr, phantom, bend left=35, "\Uparrow\gamma"] & {} \arrow[r, "u_4"'] & {}
\end{tikzcd}$$

where α, β and γ are 2-cells of X, and $u_1 v_2$, w_2, u_3 and u_4 (in the left-hand situation) or $u_1 v_2$, w_2, $v_3 w_3$ and u_4 (in the right-hand situation) are reduced non-identity 1-cells of X^*, with u_1 in X. Putting $u_2 = v_2 w_2 x_2$, in the left-hand case, or $u_2 = v_2 w_2$ and $u_3 = v_3 w_3$, in the right-hand case, induces a bijection between the 4-cells $u_1|u_2|u_3|u_4$ of $\mathrm{Sq}(X)$ and the critical 3-branchings of X.

This observation generalises to establish a bijection between the $(n+1)$-cells of $\mathrm{Sq}(X)$ and the critical n-branchings of X.

4.3.3. Theorem (Squier's polygraphic resolution). *Assume that* X *is a convergent presentation of a category* $\mathcal{C}$*. There exists a unique structure of* $(\infty, 1)$*-polygraph on* $\mathrm{Sq}(X)$*, and unique unital section* ι *and right* ι*-contraction* σ *of* $\mathrm{Sq}(X)$*, that satisfy* $\iota_u = \widehat{u}$ *for every* 1*-cell* u *of* X^**, and, for every* n*-cell* $u_1|\cdots|u_n$ *of* $\mathrm{Sq}(X)$*, with* $n \geqslant 1$*, and every reduced* 1*-cell* u_{n+1} *of* X^**,*

$$\sigma_{(u_1|\cdots|u_n)u_{n+1}} = \begin{cases} u_1|\cdots|u_{n+1} & \text{if } u_1|\cdots|u_{n+1} \in \mathrm{Sq}_{n+1}(X), \\ 1_{(u_1|\cdots|u_n)u_{n+1}} & \text{if } u_n u_{n+1} \text{ is reduced.} \end{cases} \tag{4.2}$$

As a consequence, with this structure of $(\infty, 1)$*-polygraph,* $\mathrm{Sq}(X)$ *is a polygraphic resolution of* $\mathcal{C}$*.*

Proof. If (4.2) is satisfied, then the source and target maps of $\mathrm{Sq}(X)$ are imposed by the first case, and the definition of an ι-contraction. Indeed, writing $\underline{u} = u_1|\cdots|u_{n-1}$, we must have

$$s(u_1|\cdots|u_n) = s(\sigma_{\underline{u}u_n}) = \underline{u}u_n \star_1 \sigma_{t_1(\underline{u})u_n} \star_2 \cdots \star_{n-1} \sigma_{t_{n-1}(\underline{u})u_n}$$

$$\text{and} \quad t(u_1|\cdots|u_n) = t(\sigma_{\underline{u}u_n}) = \begin{cases} \widehat{u_1u_2} & \text{if } n=2, \\ \sigma_{s(\underline{u})u_n} & \text{otherwise.} \end{cases}$$

Then one proves, using the definition of the source and target of an ι-contraction, that these source and target maps satisfy the globular relations. Next, according to Lemma 4.2.5, it is necessary and sufficient to define σ on the ι-essential and σ-essential cells of $\mathrm{Sq}(X)^\top$.

The ι-essential 1-cells are the u_1u_2, where u_1 is a 1-cell of X, u_2 is a reduced 1-cell of X^*, and u_1u_2 is not reduced. If $u_1|u_2$ is a 2-cell of $\mathrm{Sq}(X)$, then (4.2) imposes $\sigma_{u_1u_2} = u_1|u_2$. Otherwise, there exists a proper factorisation $u_2 = v_2w_2$ such that $u_1|v_2$ is a 2-cell of $\mathrm{Sq}(X)$, and (4.2) reads $\sigma_{(u_1|v_2)w_2} = 1_{(u_1|v_2)w_2}$. This last equality imposes that the source and target of $\sigma_{(u_1|v_2)w_2}$ must be equal, giving the value of σ on u_1u_2:

$$\sigma_{u_1u_2} = t(\sigma_{(u_1|v_2)w_2}) = s(\sigma_{(u_1|v_2)w_2}) = (u_1|v_2)w_2 \star_1 \sigma_{\widehat{u_1v_2}w_2}.$$

Now, fix $n \geqslant 2$. The σ-essential n-cells of $\mathrm{Sq}(X)^\top$ are the $\underline{u}u_{n+1}$, where $\underline{u} = u_1|\cdots|u_n$ is an n-cell of $\mathrm{Sq}(X)$, and u_{n+1} is a reduced 1-cell of X^*. We distinguish three cases. First, if $\underline{u}|u_{n+1}$ is an $(n+1)$-cell of $\mathrm{Sq}(X)$, then (4.2) imposes $\sigma_{\underline{u}u_{n+1}} = \underline{u}|u_{n+1}$. Second, if u_nu_{n+1} is reduced, then (4.2) gives $\sigma_{\underline{u}u_{n+1}} = 1_{\underline{u}u_{n+1}}$. Otherwise, there exists a proper factorisation $u_{n+1} = v_{n+1}w_{n+1}$ such that $\underline{u}|v_{n+1}$ is an $(n+1)$-cell of $\mathrm{Sq}(X)$. In that case, (4.2) implies that the source and the target of $\sigma_{(\underline{u}|v_{n+1})w_{n+1}}$ are equal. On the one hand, we have

$$s(\sigma_{(\underline{u}|v_{n+1})w_{n+1}}) = (\underline{u}|v_{n+1})w_{n+1} \star_1 \sigma_{t_1(\underline{u}|v_{n+1})w_{n+1}} \star_2 \cdots \star_n \sigma_{t_n(\underline{u}|v_{n+1})w_{n+1}}.$$

And, on the other hand, we obtain

$$\begin{aligned} t(\sigma_{(\underline{u}|v_{n+1})w_{n+1}}) &= \sigma_{s(\underline{u}|v_{n+1})w_{n+1}} = \sigma_{s(\sigma_{\underline{u}v_{n+1}})w_{n+1}} \\ &= \sigma_{\underline{u}u_{n+1} \star_1 \sigma_{t_1(\underline{u})v_{n+1}} w_{n+1} \star_2 \cdots \star_n \sigma_{t_n(\underline{u})v_{n+1}} w_{n+1}}. \end{aligned}$$

Using the compatibility of σ with the compositions $\star_1, \ldots, \star_n$, we develop the latter expression, by induction on n, to obtain a composite $(n+1)$-cell containing $\sigma_{\underline{u}u_{n+1}}$, $\sigma_{\sigma_{t_n(\underline{u})v_{n+1}} w_{n+1}}$, and lower dimensional invertible cells. Thus, we obtain a relation between two composite $(n+1)$-cells that defines $\sigma_{\underline{u}u_{n+1}}$ in terms of the other involved cells.

Finally, we apply Theorem 4.2.6 to conclude that $\mathrm{Sq}(X)$ is a polygraphic resolution of $\mathcal{C}$. $\square$

4.3.4. Interpretation in the reduced case. Assume that X is a reduced convergent 2-polygraph, and let us examine the first dimensions of $\mathrm{Sq}(X)$. The 2-cells $x|u$ of $\mathrm{Sq}(X)$, for x a 1-cell of X and u a reduced 1-cell of X^* such that xu is not reduced, have the shape

$$xu \overset{x|u}{\Longrightarrow} \widehat{xu}.$$

The ι-contraction σ is given, on a 1-cell xu of X^*, with $x \in X_1$ and u reduced, by

$$\sigma_{xu} = \begin{cases} x|u & \text{if } x|u \in \mathrm{Sq}_2(X), \\ 1_{xu} & \text{if } xu \text{ is reduced,} \\ (x|v)w \star_1 \sigma_{\widehat{xv}w} & \text{if } u = vw \text{ with } x|v \in \mathrm{Sq}_2(X). \end{cases}$$

On more general 1-cells, σ is defined by the fact that it is a right ι-contraction:

$$\sigma_{uv} = u\sigma_v \star_1 \sigma_{u\widehat{v}}.$$

By construction, the 3-cells of $\mathrm{Sq}(X)$ have the shape

$$\begin{tikzcd}
& \widehat{xu}v \arrow[dr, Rightarrow, "\sigma_{\widehat{xu}v}"] & \\
xuv \arrow[ur, Rightarrow, "(x|u)v"] \arrow[rr, Rightarrow, "\sigma_{xuv}"'] \arrow[rr, phantom, bend left=25, "\Downarrow x|u|v"] & & \widehat{xuv}
\end{tikzcd}$$

The ι-contraction σ is defined, on the 2-cells $(x|u)v$ by (4.2). The simple cases are $\sigma_{(x|u)v} = x|u|v$, if the latter belongs to $\mathrm{Sq}(X)$, and $\sigma_{(x|u)v} = 1_{(x|u)v}$ if uv is reduced. The more complicated case is the definition of $\sigma_{(x|u)vw}$ when $x|u|v$ belongs to $\mathrm{Sq}(X)$. In this situation, the relation $\sigma_{(x|u|v)w} = 1_{(x|u|v)w}$ implies $s(\sigma_{(x|u|v)w}) = t(\sigma_{(x|u|v)w})$, which develops into

$$\begin{tikzcd}
\widehat{xu}vw \arrow[r, Rightarrow, "\sigma_{\widehat{xu}v}w"] \arrow[dr, phantom, "\Downarrow (x|u|v)w"] & \widehat{xuv}w \arrow[d, Rightarrow, bend left, "\sigma_{\widehat{xuv}w}"] \arrow[d, phantom, "\Downarrow \sigma_{\sigma_{xuv}w}"'] \\
xuvw \arrow[u, Rightarrow, bend left, "(x|u)vw"] \arrow[ur, Rightarrow, "\sigma_{xuv}w"] \arrow[r, Rightarrow, "\sigma_{xuvw}"'] & \widehat{xuvw}
\end{tikzcd}
=
\begin{tikzcd}
\widehat{xu}vw \arrow[r, Rightarrow, "\sigma_{\widehat{xu}v}w"] \arrow[dr, Rightarrow, "\sigma_{\widehat{xu}vw}"] \arrow[dr, phantom, bend left, "\sigma_{\sigma_{\widehat{xu}v}w}\Downarrow"] & \widehat{xuv}w \arrow[d, Rightarrow, bend left, "\sigma_{\widehat{xuv}w}"] \\
xuvw \arrow[u, Rightarrow, bend left, "(x|u)vw"] \arrow[ur, phantom, "\Downarrow \sigma_{(x|u)vw}"] \arrow[r, Rightarrow, "\sigma_{xuvw}"'] & \widehat{xuvw}
\end{tikzcd}$$

Finally, the 4-cells of $\mathrm{Sq}(X)$ have the same shape as this last defining equation, but in the case where vw is not reduced and with all proper left-factors reduced:

$$\begin{tikzcd}
\widehat{xu}vw \arrow[r, Rightarrow, "\sigma_{\widehat{xu}v}w"] \arrow[dr, phantom, "\Downarrow (x|u|v)w"] & \widehat{xuv}w \arrow[d, Rightarrow, bend left, "\sigma_{\widehat{xuv}w}"] \arrow[d, phantom, "\Downarrow \sigma_{\sigma_{xuv}w}"'] \\
xuvw \arrow[u, Rightarrow, bend left, "(x|u)vw"] \arrow[ur, Rightarrow, "\sigma_{xuv}w"] \arrow[r, Rightarrow, "\sigma_{xuvw}"'] & \widehat{xuvw}
\end{tikzcd}
\overset{x|u|v|w}{\Rrightarrow}
\begin{tikzcd}
\widehat{xu}vw \arrow[r, Rightarrow, "\sigma_{\widehat{xu}v}w"] \arrow[dr, Rightarrow, "\sigma_{\widehat{xu}vw}"] \arrow[dr, phantom, bend left, "\sigma_{\sigma_{\widehat{xu}v}w}\Downarrow"] & \widehat{xuv}w \arrow[d, Rightarrow, bend left, "\sigma_{\widehat{xuv}w}"] \\
xuvw \arrow[u, Rightarrow, bend left, "(x|u)vw"] \arrow[ur, phantom, "\Downarrow \sigma_{(x|u)vw}"] \arrow[r, Rightarrow, "\sigma_{xuvw}"'] & \widehat{xuvw}
\end{tikzcd}$$

4.4. Abelianisation of polygraphic resolutions

4.4.1. Free natural systems. Let $\mathcal{C}$ be a category and X be a family of 1-cells of $\mathcal{C}$. We denote by $F_{\mathcal{C}}[X]$ the free natural system on $\mathcal{C}$ generated by X, given by

$$F_{\mathcal{C}}[X] = \bigoplus_{x \in X} \mathrm{Ct}(\mathcal{C})(x, -).$$

Fix an $(\infty,1)$-polygraph X presenting $\mathcal{C}$. We consider:

(i) The free natural system $F_{\mathcal{C}}[X_0]$ generated by the 1-cells 1_x, for $x \in X_0$. If a is a 1-cell of $\mathcal{C}$, then $F_{\mathcal{C}}[X_0]_a$ is the free abelian group generated by the pairs (b,c) of 1-cells of $\mathcal{C}$ such that $t(b) = s(c) = x$ and $bc = a$.

(ii) For every natural number $n \geqslant 1$, the free natural system $F_{\mathcal{C}}[X_n]$ is generated by one copy of the 1-cell $\overline{x}$ of $\mathcal{C}$ for each n-cell x of X. If a is a 1-cell of $\mathcal{C}$, then $F_{\mathcal{C}}[X_n]_a$ is the free abelian group generated by the triples (b,x,c), denoted by $b[x]c$, made of an n-cell x of X, and 1-cells b and c of $\mathcal{C}$, such that the composite $b\overline{x}c$ exists in $\mathcal{C}$ and is equal to a.

The mapping of every 1-cell x of X to the element $[x]$ of $F_{\mathcal{C}}[X_1]_{\overline{x}}$ is extended into a derivation of X_1^* into $F_{\mathcal{C}}[X_1]$ by putting

$$[1_u] = 0 \qquad \text{and} \qquad [uv] \;=\; [u]\overline{v} + \overline{u}[v].$$

Here, the natural system $F_{\mathcal{C}}[X_1]$ on $\mathcal{C}$ is seen as a natural system on X_1^* by composition with the canonical projection $X_1^* \twoheadrightarrow \overline{X}$. Then, for $n > 1$, the mapping of every n-cell x of X to the element $[x]$ of $F_{\mathcal{C}}[X_n]_{\overline{x}}$ is extended to associate to every n-cell a of $X^\top$ the element $[a]$ of $F_{\mathcal{C}}[X_n]_{\overline{a}}$, defined by induction on the size of a as follows:

$$[1_a] \;=\; 0 \qquad\qquad [a^-] \;=\; -[a] \qquad\qquad [a \star_k b] \;=\; \begin{cases} [a]\overline{b} + \overline{a}[b] & \text{if } k = 0,\\ [a] + [b] & \text{otherwise.}\end{cases}$$

4.4.2. Abelianisation of $(\infty,1)$-polygraphs. Let X be an $(\infty,1)$-polygraph. We denote by $F_{\overline{X}}[X]$ the complex

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\;\varepsilon\;} F_{\overline{X}}[X_0] \xleftarrow{\;\delta_1\;} \cdots \longleftarrow F_{\overline{X}}[X_{n-1}] \xleftarrow{\;\delta_n\;} F_{\overline{X}}[X_n] \longleftarrow \cdots$$

of natural systems on $\overline{X}$, whose boundary maps are defined as follows. The augmentation morphism ε is defined, on every pair (a,b) of composable 1-cells of $\overline{X}$, by

$$\varepsilon(a,b) = 1.$$

For $n \geqslant 1$, the morphism δ_n of natural systems on $\overline{X}$ is given, on the generator $[x]$ corresponding to an n-cell x of X, by:

$$\delta_n[x] \;=\; \begin{cases} (\overline{x},1) - (1,\overline{x}) & \text{if } n = 1,\\ [s(x)] - [t(x)] & \text{otherwise.}\end{cases}$$

By induction on the size of cells of $X^\top$, we prove, for every n-cell a in $X^\top$, with $n \geqslant 1$, that

$$\delta_n[a] \;=\; \begin{cases} (\overline{a},1) - (1,\overline{a}) & \text{if } n = 1,\\ [s(a)] - [t(a)] & \text{otherwise.}\end{cases}$$

As a consequence, we have $\varepsilon\delta_1 = 0$ and $\delta_n\delta_{n+1} = 0$, for every $n \geqslant 1$, proving that $F_{\overline{X}}[X]$ is indeed a chain complex.

4.4.3. Theorem. *If X is a polygraphic resolution of a category $\mathcal{C}$, then $F_{\mathcal{C}}[X]$ is a free resolution of the constant natural system $\mathbb{Z}$ on $\mathcal{C}$.*

Proof. Let ι be a unital section of X. We write $\widehat{a}$ for the image of a 1-cell a of $\mathcal{C}$ through ι. By Theorem 4.2.6, X admits a right ι-contraction σ. We denote by σ_n, for every integer $n \geqslant -1$, the following families of morphisms of groups, indexed by a 1-cell a of $\mathcal{C}$:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\sigma_{-1}} & F_{\mathcal{C}}[X_0]_a \\ 1 & \longmapsto & (a,1) \end{array} \qquad \begin{array}{ccc} F_{\mathcal{C}}[X_0]_a & \xrightarrow{\sigma_0} & F_{\mathcal{C}}[X_1]_a \\ (b,c) & \longmapsto & b[\widehat{c}] \end{array} \qquad \begin{array}{ccc} F_{\mathcal{C}}[X_n]_a & \xrightarrow{\sigma_n} & F_{\mathcal{C}}[X_{n+1}]_a \\ b[x]c & \longmapsto & b[\sigma_{x\widehat{c}}] \end{array}$$

First, by induction on the size of the n-cells of $X^\top$, using the properties of a right ι-contraction, we prove that

$$\sigma_n(b[a]c) = b[\sigma_{a\widehat{c}}]$$

holds for every $n \geqslant 1$, every n-cell a of $X^\top$, and all 1-cells b and c of $\mathcal{C}$ such that $b\overline{a}c$ exists. Then, it is straightforward to check that $(\sigma_n)_{n\geqslant 1}$ is a contracting homotopy for $F_{\mathcal{C}}[X]$. □

4.4.4. Homological syzygies. Fix an $(\infty,1)$-polygraph X. For every $n \geqslant 1$, the kernel of the differential δ_n is denoted by $h_n(X)$, and its elements are called the *homological* n*-syzygies of* X. As a consequence of Theorems 4.3.3 and 4.4.3, we obtain

4.4.5. Proposition. *Let $\mathcal{C}$ be a category, and X be a convergent presentation of $\mathcal{C}$. Then, for every $n \geqslant 2$, the natural system $h_n(X)$ of homological n-syzygies of X is generated by the elements*

$$\delta_n[u_1|\cdots|u_n] = [u_1|\cdots|u_{n-1}]\overline{u}_n + [\sigma_{t(u_1|\cdots|u_{n-1})u_n}] - [\sigma_{s(u_1|\cdots|u_{n-1})u_n}],$$

where $u_1|\cdots|u_n$ ranges over the n-cells of $\mathrm{Sq}(X)$*, and σ is the right ι-contraction associated to* $\mathrm{Sq}(X)$.

Homological syzygies generalise identities among relations in all dimensions, in the following sense:

4.4.6. Theorem. *For every 2-polygraph X, the natural systems of homological 2-syzygies and of identities among relations of X are isomorphic.*

Proof. First, we prove that, for every closed 2-cell f of $X^\top$, we have $[f] = 0$ in $F_{\overline{X}}[X_2]$ if, and only if, $\lfloor f \rfloor = 0$ holds in $\Pi(X)$, see [99, Lemma 5.6.3]. Next, we check that, for every element a in $h_2(X)$, there exists a closed 2-cell $\widetilde{a}$ in $X^\top$ such that $a = [\widetilde{a}]$ holds, see [99, Lemma 5.6.4]. Finally, we put

$$\Phi\big(\lfloor f \rfloor\big) = [f] \qquad \text{and} \qquad \Psi(a) = \lfloor \widetilde{a} \rfloor,$$

and we check that $\Phi : \Pi(X) \to h_2(X)$ and $\Psi : h_2(X) \to \Pi(X)$ are well-defined morphisms of natural systems over $\overline{X}$, that are inverse of one another, see [99, Theorem 5.6.5]. □

4.5. Homotopical and homological finiteness conditions

4.5.1. Higher finite derivation type. Let $\mathcal{C}$ be a category. For $n \in \overline{\mathbb{N}}$, with $n \neq 0$, we say that $\mathcal{C}$ is of *finite* n*-derivation type* (or that $\mathcal{C}$ is FDT_n for short) if it admits a polygraphic resolution X such that X_p is finite for every natural number $p \leqslant n$. In particular, $\mathcal{C}$ is FDT_1 if it is finitely generated, FDT_2 if it is finitely presented, and FDT_3 if it is of finite derivation type. By definition, for every natural number n, FDT_∞ implies FDT_n, and FDT_{n+1} implies FDT_n.

As an immediate consequence of Theorem 4.3.3, we obtain

4.5.2. Corollary. *A category with a finite convergent presentation is* FDT_∞.

4.5.3. Modules of finite homological type. Fix a category $\mathcal{C}$. A $\mathcal{C}$-*module* is a functor from $\mathcal{C}$ to the category $\mathcal{Ab}$ of abelian groups [166]. For $n \in \overline{\mathbb{N}}$, a $\mathcal{C}$-module M is of *homological type* FP_n, if it admits a partial resolution of length n by finitely generated projective $\mathcal{C}$-modules

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow \cdots \longleftarrow P_{n-1} \longleftarrow P_n \longleftarrow \cdots$$

By general homological arguments, we obtain the following characterisation of the property FP_n:

4.5.4. Lemma. *Let $\mathcal{C}$ be a category, M be a $\mathcal{C}$-module, and fix $n \in \overline{\mathbb{N}}$. The following assertions are equivalent:*

(i) *M is of homological type* FP_n*,*

(ii) *M admits a partial resolution of length n by finitely generated free $\mathcal{C}$-modules,*

(iii) *M is finitely generated and, for every natural number $k < n$ and every projective, finitely generated partial resolution of M of length k*

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow \cdots \longleftarrow P_{k-1} \xleftarrow{d_k} P_k,$$

the $\mathcal{C}$-module $\ker d_k$ *is finitely generated.*

4.5.5. Categories of finite homological type. The property for a category $\mathcal{C}$ to be of homological type FP_n is defined according to a category of modules over one of the categories in the following diagram

$$\begin{tikzcd}
 & & \mathcal{C}^o \arrow[dr, tail, "q_1"] & \\
\mathrm{Ct}(\mathcal{C}) \arrow[r, two heads, "\partial"] & \mathcal{C}^o \times \mathcal{C} \arrow[ur, two heads, bend left, "p_1"] \arrow[dr, two heads, bend right, "p_2"'] & & \mathcal{C}^\top \\
 & & \mathcal{C} \arrow[ur, tail, "q_2"'] &
\end{tikzcd}$$

where ∂ is the boundary map, p_1 and p_2 are the projections of the cartesian product, $\mathcal{C}^\top$ is the enveloping groupoid of $\mathcal{C}$, and q_1 and q_2 are the canonical inclusion morphisms. A category $\mathcal{C}$ is of *homological type*

(i) FP_n if the constant natural system $\mathbb{Z}$ is of type FP_n,

(ii) *bi*-FP_n if the $\mathcal{C}^o \times \mathcal{C}$-module $\mathbb{Z}\mathcal{C}$ is of type FP_n,

(iii) *left*-FP_n if the constant $\mathcal{C}$-module $\mathbb{Z}$ is of type FP_n,

(iv) *right*-FP_n if the constant $\mathcal{C}^o$-module $\mathbb{Z}$ is of type FP_n,

(v) *top*-FP_n if the constant $\mathcal{C}^\top$-module $\mathbb{Z}$ is of type FP_n.

Using the fact that the property FP_n is preserved by left Kan extensions [99, Lemma 5.1.4], these finiteness homological properties of categories are related by the following implications [99, Proposition 5.2.4]:

$$
\begin{tikzcd}
 & & \text{right-FP}_n \arrow[dr, Rightarrow] & \\
\text{FP}_n \arrow[r, Rightarrow] & \text{bi-FP}_n \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{top-FP}_n. \\
 & & \text{left-FP}_n \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

If $\mathcal{C}$ is a groupoid, all of these implications are equivalences [99, Proposition 5.2.6], but it is not the case in general. Indeed, Cohen constructed in [55] a right-FP_∞ monoid which is not left-FP_1: thus, top-FP_n, left-FP_n and right-FP_n are not equivalent in general. Moreover, monoids with a finite convergent presentation are left-FP_∞ and right-FP_∞, see [6, 192, 127], but there exists a finitely presented monoid that is left-FP_∞ and right-FP_∞, but does not satisfy the homological finiteness condition FHT introduced by Pride and Wang [128]; since the properties FHT and bi-FP_3 are equivalent [129], it follows that left-FP_n and right-FP_n do not imply bi-FP_n in general. We conjecture that bi-FP_n does not imply FP_n either.

As a consequence of Theorems 4.3.3 and 4.4.3, we obtain the following implication. It generalises [57, Theorem 3.2] and [133, Theorem 3], see also [181], stating that, if a monoid is of finite derivation type, then it is FP_3. It also generalises Squier's homological theorem [192, Theorem 4.1], that says that a monoid admitting a finite convergent presentation is FP_3.

4.5.6. Theorem. *Let $\mathcal{C}$ be a category. For every $n \in \overline{\mathbb{N}}$, if $\mathcal{C}$ is FDT_n, then it is FP_n. In particular, if $\mathcal{C}$ admits a finite convergent presentation, then it is FP_∞.*

Finally, Theorems 3.6.6 and 4.4.6, and Lemma 4.5.4 give the following equivalences:

4.5.7. Theorem. *Let $\mathcal{C}$ be a category, and X be a finite presentation of $\mathcal{C}$. The following assertions are equivalent:*

- ***(i)*** $\mathcal{C}$ *is* FP_3.
- ***(ii)*** $\mathcal{C}$ *is* FDT_{ab}.
- ***(iii)*** $h_2(X)$ *is finitely generated.*
- ***(iv)*** $\Pi(X)$ *is finitely generated.*

4.6. Examples of polygraphic and free resolutions

4.6.1. Example. Consider the monoid M presented by the 2-polygraph

$$X = \left(a, b, t \;\middle|\; at^n b \overset{\alpha_n}{\Longrightarrow} 1 \,,\; n \in \mathbb{N} \right).$$

Being finitely generated, M is FP_1, but it is not FP_2. Indeed, first, observe that X is reduced and convergent, with no critical branching. As a consequence, Theorem 4.3.3 implies that X, seen as an $(\infty, 1)$-polygraph, is a polygraphic resolution of M. Next, use Theorem 4.4.3 to obtain the free resolution $F_M[X]$

$$0 \longleftarrow \mathbb{Z} \overset{\varepsilon}{\longleftarrow} F_M[*] \overset{\delta_1}{\longleftarrow} F_M[X_1] \overset{\delta_2}{\longleftarrow} F_M[X_2] \longleftarrow 0$$

Hence $\ker \delta_2 = 0$, so that $\ker \delta_1 \simeq F_M[X_2]$. So $\ker \delta_1$ is not finitely generated, and Lemma 4.5.4 concludes that M is not FP_2.

4.6.2. The reduced standard polygraphic resolution. Let $\mathcal{C}$ be a category. To simplify the example, assume that, if a composite 1-cell xy of $\mathcal{C}$ is an identity, then so are x and y. The *reduced standard presentation of* $\mathcal{C}$ is the 2-polygraph $\overline{\mathrm{Std}}_2(\mathcal{C})$ with the same 0-cells as $\mathcal{C}$, one 1-cell $\widehat{a}$ for each non-identity 1-cell a of $\mathcal{C}$, and one 2-cell $a|b : \widehat{a}\widehat{b} \Rightarrow \widehat{ab}$ for each pair (a, b) of composable non-identity 1-cells in $\mathcal{C}$. Without the simplifying hypothesis on $\mathcal{C}$, the target of $a|b$ is replaced by 1_x if $ab = 1_x$.

The 2-polygraph $\overline{\mathrm{Std}}_2(\mathcal{C})$ is reduced and convergent, and applying Theorem 4.3.3 extends it into a polygraphic resolution of $\mathcal{C}$, denoted by $\overline{\mathrm{Std}}(\mathcal{C})$ and called the *reduced standard polygraphic resolution of* $\mathcal{C}$. For $n \geqslant 2$, the n-cells of $\mathrm{Std}(\mathcal{C})$ are the $a_1|\cdots|a_n$, such that each a_i is a non-identity 1-cell of $\mathcal{C}$ and each (a_i, a_{i+1}) is composable.

The source and target of the 3-cells of $\overline{\mathrm{Std}}(\mathcal{C})$ are given by

$$\begin{tikzcd}
 & \widehat{ab}\widehat{c} \arrow[dr, Rightarrow, "ab|c"] & \\
\widehat{a}\widehat{b}\widehat{c} \arrow[ur, Rightarrow, "(a|b)\widehat{c}"] \arrow[dr, Rightarrow, "\widehat{a}(b|c)"'] \arrow[rr, phantom, "\Downarrow a|b|c"] & & \widehat{abc} \\
 & \widehat{a}\widehat{bc} \arrow[ur, Rightarrow, "a|bc"'] &
\end{tikzcd}$$

and the ones of its 4-cells, with the arrows of 3-cells removed, by

$$\begin{tikzcd}
 & \widehat{ab}\widehat{c}\widehat{d} \arrow[rr, Rightarrow, "(ab|c)\widehat{d}"] & & \widehat{abc}\widehat{d} \arrow[dr, Rightarrow, "abc|d"] & \\
 & & (a|b|c)\widehat{d} \quad (a|bc)\widehat{d} & & \\
\widehat{a}\widehat{b}\widehat{c}\widehat{d} \arrow[uur, Rightarrow, "(a|b)\widehat{c}\widehat{d}"] \arrow[rr, Rightarrow, "\widehat{a}(b|c)\widehat{d}"] \arrow[ddr, Rightarrow, "\widehat{a}\widehat{b}(c|d)"'] & & \widehat{a}\widehat{bc}\widehat{d} \arrow[uur, Rightarrow] \arrow[ddr, Rightarrow] \arrow[r, phantom, "a|bc|d"] & & \widehat{abcd} \\
 & & \widehat{a}(b|c|d) \quad \widehat{a}(bc|d) & & \\
 & \widehat{a}\widehat{b}\widehat{cd} \arrow[rr, Rightarrow, "\widehat{a}(b|cd)"'] & & \widehat{a}\widehat{bcd} \arrow[uur, Rightarrow, "a|bcd"'] &
\end{tikzcd}$$

$$\Downarrow a|b|c|d$$

$$\begin{tikzcd}
 & \widehat{ab}\widehat{c}\widehat{d} \arrow[rr, Rightarrow, "(ab|c)\widehat{d}"] \arrow[ddr, Rightarrow, "\widehat{ab}(c|d)"] & & \widehat{abc}\widehat{d} \arrow[ddr, Rightarrow, "abc|d"] & \\
 & & & ab|c|d & \\
\widehat{a}\widehat{b}\widehat{c}\widehat{d} \arrow[uur, Rightarrow, "(a|b)\widehat{c}\widehat{d}"] \arrow[ddr, Rightarrow, "\widehat{a}\widehat{b}(c|d)"'] \arrow[rr, phantom, "="] & & \widehat{ab}\widehat{cd} \arrow[rr, Rightarrow, "ab|cd"] & & \widehat{abcd}. \\
 & & & a|b|cd & \\
 & \widehat{a}\widehat{b}\widehat{cd} \arrow[rr, Rightarrow, "\widehat{a}(b|cd)"'] \arrow[uur, Rightarrow, "(a|b)\widehat{cd}"] & & \widehat{a}\widehat{bcd} \arrow[uur, Rightarrow, "a|bcd"'] &
\end{tikzcd}$$

For n-cells, $n \geqslant 2$, we prove, by induction on n, that the source and target of n-cells are composites of the $(n-1)$-cells

$$d_i(a_1|\cdots|a_n) = \begin{cases} \widehat{a}_1(a_2|\cdots|a_n) & \text{if } i = 0, \\ a_1|\cdots|a_i a_{i+1}|\cdots|a_n & \text{if } 1 \leqslant i \leqslant n-1, \\ (a_1|\cdots|a_{n-1})\widehat{a}_n & \text{if } i = n. \end{cases}$$

with k-cells, for $1 < k < n-1$. More precisely, the source of $a_1|\cdots|a_n$ contains one copy of each $d_i(a_1|\cdots|a_n)$ for $n-i$ even, and its target, one copy of each $d_i(a_1|\cdots|a_n)$ for $n-i$ odd.

Theorem 4.4.3 applied to $\overline{\mathrm{Std}}(\mathcal{C})$ gives a free resolution

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \varepsilon\ } F_{\mathcal{C}}[\mathcal{C}_0] \longleftarrow \ \cdots \ \longleftarrow F_{\mathcal{C}}[\overline{\mathrm{Std}}_{n-1}(\mathcal{C})] \xleftarrow{\ \delta_n\ } F_{\mathcal{C}}[\overline{\mathrm{Std}}_n(\mathcal{C})] \longleftarrow \ \cdots$$

with differential

$$\delta_n[a_1|\cdots|a_n] = \sum_{i=0}^{n} (-1)^{n-i}[d_i(a_1|\cdots|a_n)].$$

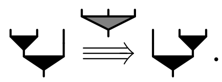
Note that, by construction, $[d_0(a_1|\cdots|a_n)] = [\widehat{a}_1(a_2|\cdots|a_n)] = a_1[a_2|\cdots|a_n]$ and, symmetrically, $[d_n(a_1|\cdots|a_n)] = [a_1|\cdots|a_{n-1}]a_n$.

4.6.3. The associative polygraphic resolution. Consider the monoid A from Example 2.6.6, with one non-trivial element a and product $a^2 = a$, presented by

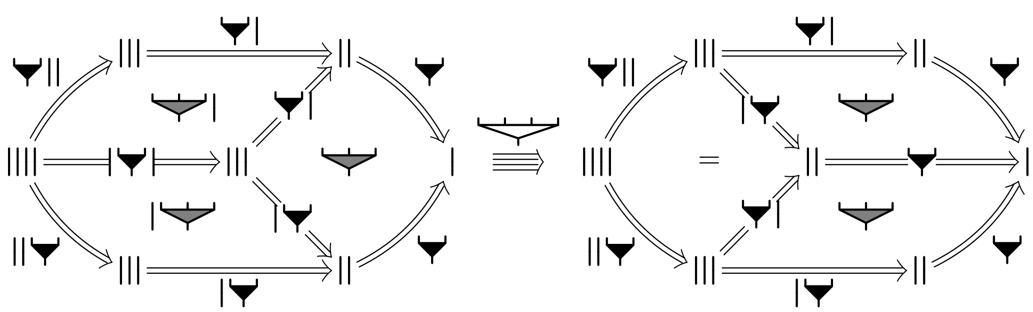
$$\mathrm{As}_2 = (a_0 \,|\, a_1 : a_0 \to a_0 \,|\, a_2 : a_1 a_1 \Rightarrow a_1)\,.$$

This 2-polygraph is reduced and convergent, with one critical n-branching for every $n \geqslant 2$. Thus, the reduced standard polygraphic resolution $\mathrm{As}_\infty = \mathrm{Sq}(\mathrm{As}_2)$ of A, given by Theorem 4.3.3, has one n-cell a_n for each $n \geqslant 0$, corresponding to the product $a|\cdots|a$ of n copies of a. Hence A is FDT_∞. The 3-cell a_3 of As_∞, as already seen in Example 2.6.6, is, given in classical notation and in string diagrams respectively,

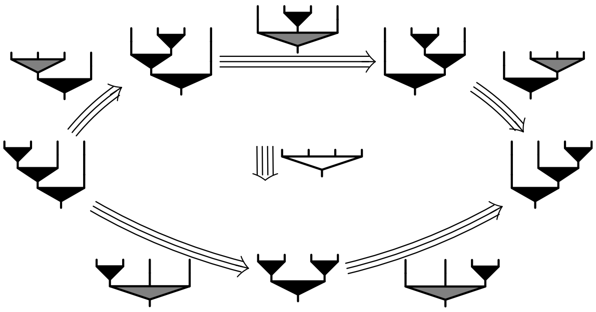
$$a_2 a_1 \star_1 a_2 \overset{a_3}{\Rrightarrow} a_1 a_2 \star_1 a_2$$



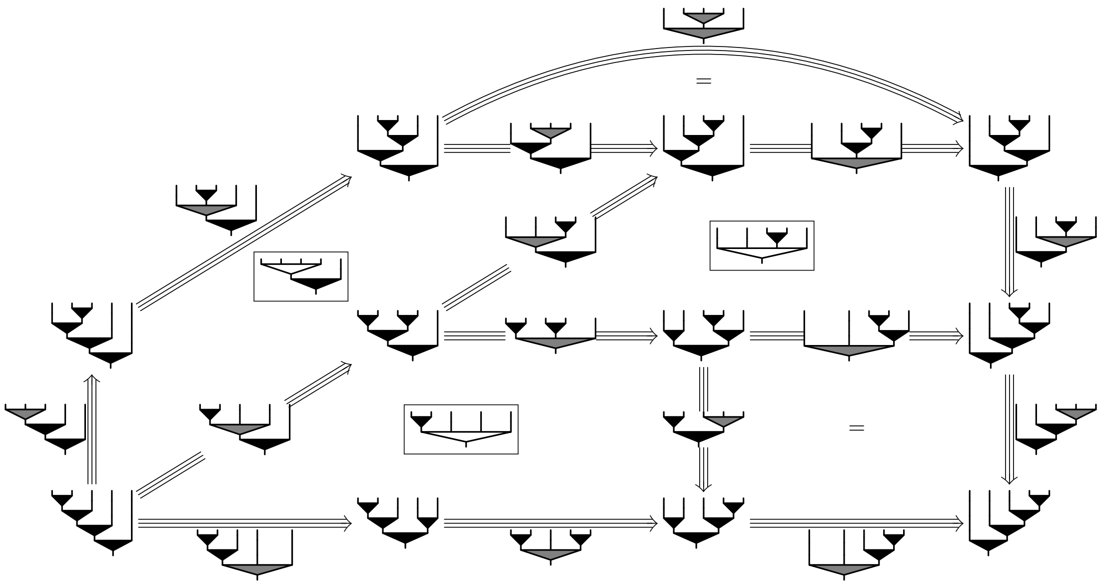
The 4-cell a_4 of As_∞ is



which, contracting by one dimension, can also be pictured as Mac Lane's pentagon, or Stasheff's polytope K_4:

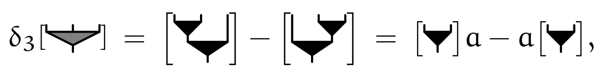


Finally, the 5-cell a_5 of As_∞ has the shape of Stasheff's polytope K_5, its source being

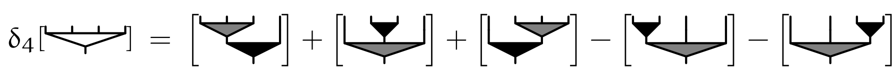


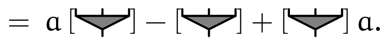
and its target being given by a symmetric composite 4-cell, see [99, §6.1]. Theorem 4.4.3, applied to As_∞, yields a resolution

of $\mathbb{Z}$ by free natural systems on A. Hence, A is FP_∞. Computing this differential on each n-cell of As_∞ gives generators of the natural systems of homological n-syzygies of As_∞. For example, $h_2(As)$ is generated by

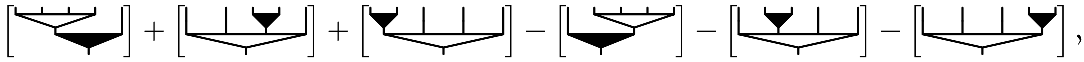


while $h_3(As)$ is generated by

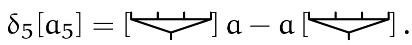




Similarly, $h_4(\mathrm{As})$ is generated by $\delta_5[a_5]$, which is equal, by definition, to



and reduces to



4.6.4. The category Epi**.** We denote by Epi the subcategory of the simplicial category whose objects are the natural numbers and whose morphisms from m to n are the monotone surjections from $\{0, \dots, m\}$ to $\{0, \dots, n\}$. This category, studied in [143] where it is written Δ^{epi}, admits a presentation by the 2-polygraph X with the natural numbers as 0-cells, with one 1-cell $x_i^n : n+1 \to n$ for all natural numbers $1 \leqslant i \leqslant n$, and one 2-cell

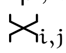
$$\begin{tikzcd}[column sep=small] & n+1 \arrow[dr, "x_j^n"] & \\ n+2 \arrow[ur, "x_i^{n+1}"] \arrow[dr, "x_{j+1}^{n+1}"'] & \Downarrow x_{i,j}^n & n \\ & n+1 \arrow[ur, "x_i^n"'] & \end{tikzcd}$$

for all natural numbers $0 \leqslant i \leqslant j \leqslant n+1$. The 1-cell x_i^n represents the map

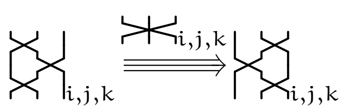
$$x_i^n(j) \;=\; \begin{cases} j & \text{if } j \leqslant i, \\ j-1 & \text{if } j > i. \end{cases}$$

Thereafter, we drop the exponents of the 1-cells and 2-cells of X, simply writing x_i and $x_{i,j}$.

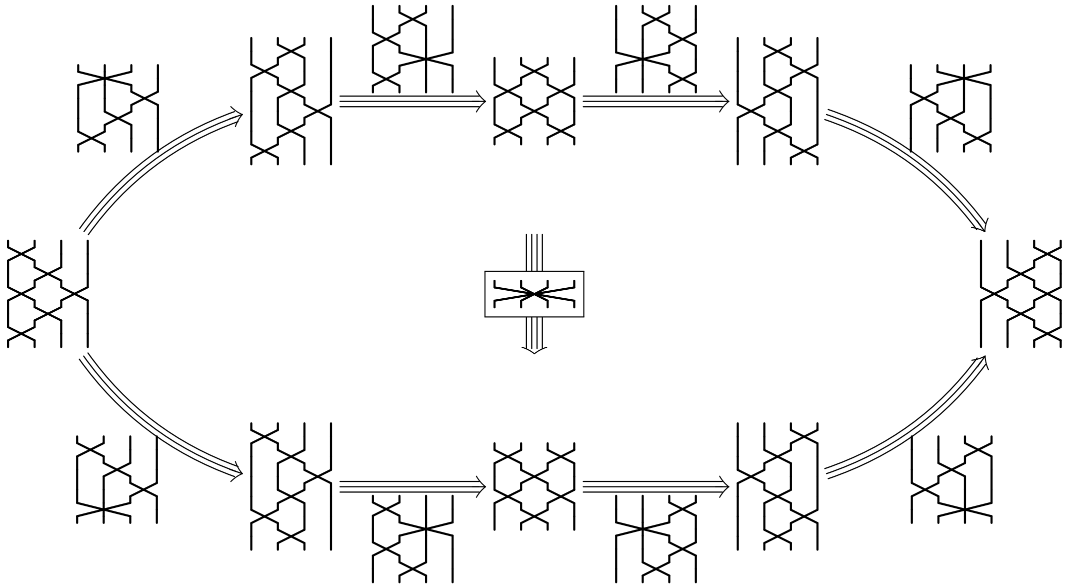
The 2-polygraph X is convergent. Indeed, for termination, given a 1-cell $u = x_{i_1} \dots x_{i_k}$ of X^*, we define the natural number $\nu(u)$ as the number of pairs (i_p, i_q) such that $i_p \leqslant i_q$, with $1 \leqslant p < q \leqslant k$. In particular, we have $\nu(x_i x_j) = 1$ and $\nu(x_{j+1} x_i) = 0$ when $i \leqslant j$, giving $\nu(s(x_{i,j})) > \nu(t(x_{i,j}))$. Moreover, we have $\nu(wuw') > \nu(wvw')$ when $\nu(u) > \nu(v)$ holds. Thus, for every non-identity 2-cell $a : u \Rightarrow v$ of X^*, the strict inequality $\nu(u) > \nu(v)$ is satisfied, giving termination. Moreover, the 2-polygraph X has one critical branching $(x_{i,j} x_k, x_i x_{j,k})$ for all possible $0 \leqslant i \leqslant j \leqslant k \leqslant n+2$, which is confluent.

Theorem 4.3.3, applied to X, gives a polygraphic resolution $\mathrm{Sq}(X)$ of Epi, whose 3-cells are given, in classical notation and in string diagrams (with $x_i =$  and $x_{i,j} =$ ) respectively, by

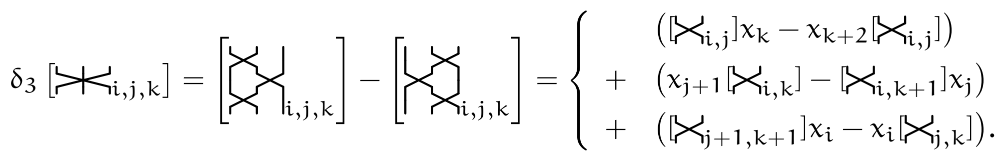
$$\begin{tikzcd}[column sep=small] & x_{j+1}x_ix_k \arrow[rr, Rightarrow, "x_{j+1}x_{i,k}"] & & x_{j+1}x_{k+1}x_i \arrow[dr, Rightarrow, "x_{j+1,k+1}x_i"] & \\ x_ix_jx_k \arrow[ur, Rightarrow, "x_{i,j}x_k"] \arrow[dr, Rightarrow, "x_ix_{j,k}"'] & & \Downarrow x_{i,j,k} & & x_{k+2}x_{j+1}x_i \\ & x_ix_{k+1}x_j \arrow[rr, Rightarrow, "x_{i,k+1}x_j"'] & & x_{k+2}x_ix_j \arrow[ur, Rightarrow, "x_{k+2}x_{i,j}"'] & \end{tikzcd}$$



The $(\infty, 1)$-polygraph $\mathrm{Sq}(X)$ has one 4-cell $x_{i,j,k,l}$ for all possible $0 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n+3$, given in string diagrams and omitting the subscripts, by



Then, Theorem 4.4.3 gives, in particular, generators for the natural systems of homological n-syzygies of X. For example, $h_2(X)$ is generated by the elements



Chapter 5
Coherent presentations of Artin monoids

In this chapter, we use the following notation: if X is a $(3,1)$-polygraph, we denote by $\widetilde{X}$ the $(2,1)$-category presented by X, i.e. the quotient $X_2^{\top}/X_3$, while $\overline{X}$ still denotes the 1-category presented by X, i.e. the quotient X_1^*/X_2. If S is a set, and s and t are elements of S, we denote by $\langle st\rangle^n$ the element of S^* defined by $\langle st\rangle^0 = 1$ and $\langle st\rangle^{n+1} = s\,\langle ts\rangle^n$.

5.1. Introduction

5.1.1. Context. A *Coxeter group* is a group W that admits a presentation by a finite set S of generators, submitted to the relations $s^2 = 1$ for every s in S, and at most one *braid relation*

$$\langle st\rangle^{m_{st}} = \langle ts\rangle^{m_{st}}$$

for each non-ordered pair (s,t) of distinct elements of S, with $m_{st} \in \mathbb{N} \setminus \{0,1\}$. A given Coxeter group may admit several generating sets that fit the given scheme, but, here, we always assume that such a set S has been fixed. Forgetting about the involutive character of the generators and keeping only the braid relations, one gets Artin's presentation of the Artin monoid $B^+(W)$. For example, if W is the group S_4 of permutations of $\{1,2,3,4\}$, then S consists of the elementary transpositions $r = (1\ 2)$, $s = (2\ 3)$ and $t = (3\ 4)$, and the associated Artin monoid is the monoid B_4^+ of positive braids on four strands, with generators r, s and t satisfying the relations $rsr = srs$, $rt = tr$ and $sts = tst$. The aim of [78] was to push further Artin's presentation and study the relations between the braid relations to obtain a coherent presentation of the Artin monoid $B^+(W)$ for every Coxeter group W.

In the case of the braid monoid B_4^+ on four strands, Deligne notes in §1.3 of [67] that Artin's presentation can be extended into a coherent presentation by taking one 3-cell whose boundary consists of the graph of reduced expressions of the element of maximal length of S_4. Such a graph can be considered for any element w of W: the vertices are the reduced expressions of w, and two such expressions are linked by an edge if one is obtained from the other by the application of a braid relation. Tits proved that the fundamental group of the graph of reduced expressions of w is generated by two types of loops in the graph, the most interesting ones being associated to finite parabolic subgroups of rank 3 of W, see [201, Proposition 4] or [186, Theorem 2.17]. Actually, for the purpose of finding a coherent presentation of $B^+(W)$, the generators of the first type are degenerate, and part of the generators of the second type are superfluous.

The main motivation of Deligne in [67] was to study a weak form of action of Artin monoids on categories. Given a monoid M and a category $\mathcal{C}$, a strict action of M of $\mathcal{C}$ would be specified by an endofunctor $F(x)$ of $\mathcal{C}$ for every element of M, such that the relations $F(x)F(y) = F(xy)$ and $F(1) = 1_{\mathcal{C}}$ hold. To define a weak action of M on $\mathcal{C}$, in Deligne's sense, one would replace these last equalities by natural isomorphisms, that themselves satisfy coherence relations. In the case where M is an Artin monoid $B^+(W)$, Deligne proved that this definition of action of $B^+(W)$ can be reformulated in a more compact way, using only an endofunctor $F(w)$ for each w of W and not of $B^+(W)$, see [67, Theorem 1.5]. The description of this alternative definition is based on an alternative presentation of Artin monoids, that we call Garside's presentation here, because it originates in Garside's study of the normal forms of braids in [77].

5.1.2. Summary. The relationship between coherent presentations and Deligne's weak actions is explained in §5.2, using the homotopical setting of the canonical model structure on 2-categories given by Lack in [131, 132]. First, we establish the following equivalence between coherent presentations of categories and some of their cofibrant approximations:

Theorem 5.2.3. *Let $\mathcal{C}$ be a category, and X be a $(3,1)$-polygraph that presents $\mathcal{C}$. Then X is a coherent presentation of $\mathcal{C}$ if, and only if, the $(2,1)$-category $\widetilde{X}$ presented by X is a cofibrant approximation of $\mathcal{C}$.*

Then, we study the category $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C},\mathcal{D})$ of pseudofunctors from a 2-category $\mathcal{C}$ into a 2-category $\mathcal{D}$: pseudofunctors are a weakened version of 2-functors, that generalise actions of monoids on categories, in the sense that a weak action of M on a category is a pseudofunctor from M into $\mathcal{C}at$. In representation theory, pseudofunctors have been studied under the name of 2-representations by Elgueta for 2-groups in [69], and Ganter and Kapranov in [76] for groups, while Rouquier considered in [187] the more general case of 2-representations of bicategories; in these examples, $\mathcal{D}$ is usually the 2-category of 2-vector spaces, either from Kapranov and Voevodsky [121] or from Baez and Crans [15], or of 2-Hilbert spaces [14], while Deligne takes the 2-category of categories in [67]. The main result of the section establishes that, for every coherent presentation X of a category $\mathcal{C}$, the pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ are the same as the 2-functors from the $(2,1)$-category $\widetilde{X}$ presented by X into $\mathcal{D}$:

Theorem 5.2.8. *Let $\mathcal{C}$ be a category, and X be a $(3,1)$-polygraph that presents $\mathcal{C}$. Then X is a coherent presentation of $\mathcal{C}$ if, and only if, for every 2-category $\mathcal{D}$, there is an equivalence of categories $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C},\mathcal{D}) \approx 2\mathcal{C}at(\widetilde{X},\mathcal{D})$ that is natural in $\mathcal{D}$.*

As a consequence, Deligne's Theorem 1.5 of [67] is another way of saying that a specific $(3,1)$-polygraph, denoted by $\mathrm{Gar}_3(W)$ in what follows, is a coherent presentation of $B^+(W)$ for every finite Coxeter group W.

Next, §5.3 explores a theoretical setting for transformations of $(3,1)$-polygraphs that do not change their homotopy type: two $(3,1)$-polygraphs are called Tietze-equivalent if they present isomorphic categories and equivalent $(2,1)$-categories, so that, in particular, two coherent presentations of the same category are Tietze-equivalent. We generalise to $(3,1)$-polygraphs the elementary operations called Tietze transformations, after the ones originally defined by Tietze for presentations of groups [200, 151]: in our case, they correspond to the simultaneous adjunction or elimination of an n-cell and of an $(n+1)$-cell, such as a redundant generator and a relation that defines it in terms of the other generators. The main property of Tietze transformations is given by

Theorem 5.3.3. *Two $(3,1)$-polygraphs X and Y are Tietze-equivalent if, and only if, there exists a Tietze transformation between them.*

As a consequence, if X is a coherent presentation of a category $\mathcal{C}$, and Y obtained from X by a sequence of Tietze transformations, then Y is also a coherent presentation of $\mathcal{C}$.

Tietze transformations are used in §5.4 to develop a procedure called homotopical completion-reduction, whose aim is to compute a small coherent presentation from a terminating presentation. First, homotopical completion is a composition of Knuth-Bendix's completion and of Squier's completion, that extends a terminating presentation into a coherent presentation by application of a sequence of Tietze transformations; in fact, both completions can be interleaved to optimise computation [78, §2.2.4]. Then, homotopical reduction is based on the notion of collapsible part of a $(3,1)$-polygraph X, named in analogy with Brown's work [37]: this is a subset of the cells of X that can be collapsed in such a way that the result is Tietze-equivalent to X. We obtain

Theorem 5.4.4. *Assume that X is a terminating presentation of a category $\mathcal{C}$. Then, every homotopical completion-reduction of X is a coherent presentation of $\mathcal{C}$.*

In concrete examples, when X is obtained by homotopical completion, the study of the critical 3-branchings of X usually produces a collapsible part that allows to eliminate 3-cells while preserving Tietze equivalence. Apart from the applications to Artin monoids, developed in [78] and presented in this chapter, the homotopical completion-reduction procedure has been applied to other examples, such as Artin monoids with other generating sets, or the plactic and Chinese monoids with various generating sets, in [101] and [104].

The homotopical completion-reduction procedure is used in §5.5 to obtain a coherent presentation of the Artin monoid $B^+(W)$ for every Coxeter group W. The starting presentation is called Garside's presentation of $B^+(W)$, as given in [67, §1.4.5] in the spherical case, i.e. when W is finite, and in [163, Proposition 1.1] in general. Its generators are the elements of $W_+ = W \setminus \{1\}$, and it has one 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$ for all u and v in W_+ such that $l(uv) = l(u) + l(v)$ holds, where the notation $\cdot|\cdot$ stands for the product in the free monoid over W_+, and $l(u)$ is the length of u in W. Garside's presentation is terminating but not confluent, and homotopical completion yields a coherent presentation of $B^+(W)$; the construction relies on specific arithmetic properties of Artin monoids, first observed by Garside for braid monoids in [77], generalised by Brieskorn and Saito in [36] and Deligne in [66], and summarised in [80]. Among the cells of this first coherent presentation of $B^+(W)$, we identify a collapsible part and obtain, after homotopical reduction, Garside's coherent presentation of $B^+(W)$:

Theorem 5.5.5. *Let W be a Coxeter group. Denote by $\mathrm{Gar}_3(W)$, the $(3,1)$-polygraph obtained from Garside's presentation of $B^+(W)$ by adjunction of one 3-cell*

$$\begin{tikzcd} & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\ u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "\Downarrow A_{u,v,w}"] & & uvw \\ & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] & \end{tikzcd}$$

for all u, v *and* w *of* W_+ *such that* $l(uvw) = l(u) + l(v) + l(w)$. *Then* $\mathrm{Gar}_3(W)$ *is a coherent presentation of* $B^+(W)$.

We note that $\mathrm{Gar}_3(W)$ is the same coherent presentation that corresponds to Deligne's Theorem 1.5 of [67], through the equivalence of Theorem 5.2.8. This implies that Deligne's result, only proved for W finite, is in fact true in general. In §3.3 of [78], Deligne's result is also extended in another direction, from spherical Artin monoids to the more general Garside monoids, introduced by Dehornoy and Paris to axiomatise the arithmetic properties of the formers [65, 59]. We conjecture that these two disjoint extensions of Theorem 5.5.5 can be unified again, in the even more general setting of monoids with a Garside family [60].

Finally, in §5.6, we homotopically reduce Garside's coherent presentation $\mathrm{Gar}_3(W)$ into the smaller coherent presentation $\mathrm{Art}_3(W)$ associated with Artin's presentation of the monoid $B^+(W)$. After a new collapsing, assuming that the set S of generators of W is totally ordered, the set of 3-cells of $\mathrm{Gar}_3(W)$ boils down to one 3-cell $Z_{r,s,t}$ for all elements $t > s > r$ of S such that the subgroup of W they span is finite:

Theorem 5.6.4. *Fix a Coxeter group* W*, with a totally ordered set of generators* S*. The Artin monoid* $B^+(W)$ *admits, as a coherent presentation, the* $(3,1)$*-polygraph* $\mathrm{Art}_3(W)$ *made of Artin's presentation* $\mathrm{Art}_2(W)$*, extended with one* 3*-cell* $Z_{r,s,t}$ *for all elements* $t > s > r$ *of* S *such that the subgroup* $W_{\{r,s,t\}}$ *is finite, and whose shape depends only on the Coxeter type of* $W_{\{r,s,t\}}$.

The precise shapes of these 3-cells, called the Tits-Zamolodchikov 3-cells in [78], are given at the end of the section: the classification of finite Coxeter groups of rank 3 implies that there are exactly five cases, corresponding to specific lengths of the braid relations. Theorem 5.6.4 improves Tits' Proposition 4 of [201], reducing the set of generators of the fundamental group of the graph of reduced expressions of w in W to the cells $Z_{r,s,t}$. Moreover, as a byproduct, to determine the action of an Artin monoid on a category, it suffices to attach to any generating 1-cell $s \in S$ an endofunctor $T(s)$ and to any generating 2-cell a natural isomorphism, such that these satisfy coherence relations given by the Tits-Zamolodchikov 3-cells.

5.2. Homotopical properties of coherent presentations

5.2.1. Pseudofunctors. Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, a *pseudofunctor from* $\mathcal{C}$ *to* $\mathcal{D}$ is a suitably weakened notion of 2-functor $F : \mathcal{C} \to \mathcal{D}$, with a strict compatibility with the composition $\star_1$, but only a weakened compatibility with the composition $\star_0$. This compatibility is expressed by

(i) an invertible 2-cell $F_{a,b} : F(a)F(b) \Rightarrow F(ab)$ of $\mathcal{D}$, natural in a and b, for every 0-composable pair (a, b) of 1-cells of $\mathcal{C}$,

(ii) an invertible 2-cell $F_x : 1_{F(x)} \Rightarrow F(1_x)$ of $\mathcal{D}$, for every 0-cell x of $\mathcal{C}$,

that satisfy classical monoidal coherence relations. The latter imply that, for every sequence $(a_1, \ldots, a_n)$ of 0-composable 1-cells in $\mathcal{C}$, there exists a unique invertible 2-cell

$$F_{a_1,\ldots,a_n} : F(a_1)\cdots F(a_n) \Rightarrow F(a_1 \cdots a_n)$$

in $\mathcal{D}$, built from the coherence isomorphisms of F. A 2-functor is just a pseudofunctor whose coherence 2-cells are identities. Morphisms of pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ are pseudonatural transformations, giving rise to a category $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{D})$. Its full subcategory whose objects are the strict 2-functors is written $2\mathcal{C}at(\mathcal{C}, \mathcal{D})$.

5.2.2. The canonical model structure on $2\mathcal{C}at$. We recall elements of the model structure for 2-categories established in [131, 132]. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a *weak equivalence* if it satisfies the following two conditions:

- **(i)** every 0-cell y of $\mathcal{D}$ is equivalent to some 0-cell $F(x)$, for x in $\mathcal{C}$;
- **(ii)** for all 0-cells x and x' in $\mathcal{C}$, the induced functor $F : \mathcal{C}(x, x') \to \mathcal{D}(F(x), F(x'))$ is an equivalence of categories.

Thus, $F : \mathcal{C} \to \mathcal{D}$ is a weak equivalence if, and only if, there exists a pseudofunctor $G : \mathcal{D} \to \mathcal{C}$ that is a quasi-inverse for F, i.e. such that $GF \simeq 1_{\mathcal{C}}$ and $FG \simeq 1_{\mathcal{D}}$. In particular, an equivalence of 2-categories is a weak equivalence. We say that $\mathcal{C}$ is *cofibrant* if it is 1-free (its underlying 1-category is free), and that $\mathcal{D}$ is a *cofibrant approximation of* $\mathcal{C}$ if $\mathcal{D}$ is cofibrant and if there exists a weak equivalence $\mathcal{D} \to \mathcal{C}$.

Coherent presentations of categories are closely related to their cofibrant approximations in $2\mathcal{C}at$:

5.2.3. Theorem ([78, Theorem 1.3.1]). *Let $\mathcal{C}$ be a category, and X be a $(3,1)$-polygraph that presents $\mathcal{C}$. The following assertions are equivalent:*

- ***(i)*** *X is a coherent presentation of $\mathcal{C}$.*
- ***(ii)*** *$\widetilde{X}$ is a cofibrant approximation of $\mathcal{C}$.*

5.2.4. The standard cofibrant approximation. For any 2-category $\mathcal{C}$, we denote by $\widehat{\mathcal{C}}$ the cofibrant 2-category with the same 0-cells as $\mathcal{C}$ and the following higher cells:

- **(i)** the 1-cells of $\widehat{\mathcal{C}}$ are freely generated by the ones of $\mathcal{C}$, with a in $\mathcal{C}$ written $\widehat{a}$ when seen as a generator of $\widehat{\mathcal{C}}$;
- **(ii)** the 2-cells from $\widehat{a}_1 \cdots \widehat{a}_m$ to $\widehat{b}_1 \cdots \widehat{b}_n$ in $\widehat{\mathcal{C}}$ are the 2-cells from $a_1 \cdots a_m$ to $b_1 \cdots b_n$ in $\mathcal{C}$, with the same compositions as in $\mathcal{C}$.

The canonical projection $\widehat{\mathcal{C}} \twoheadrightarrow \mathcal{C}$ is the identity on 0-cells and maps each generating 1-cell $\widehat{a}$ to a, and each 2-cell to itself: this is a weak equivalence, whose quasi-inverse lifts a 2-cell $f : a \Rightarrow b$ to its distinguished representative $\widehat{f} : \widehat{a} \Rightarrow \widehat{b}$. Hence, the 2-category $\widehat{\mathcal{C}}$ is a cofibrant approximation of $\mathcal{C}$, called the *standard cofibrant approximation of* $\mathcal{C}$.

When $\mathcal{C}$ is a category, the 2-category $\widehat{\mathcal{C}}$ is the 2-category presented by the standard coherent presentation of $\mathcal{C}$ given in §3.3.3. Moreover, a 2-functor from $\widehat{\mathcal{C}}$ to a 2-category $\mathcal{D}$ is the same as a pseudofunctor from $\mathcal{C}$ to $\mathcal{D}$, yielding an isomorphism $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{D}) \simeq 2\mathcal{C}at(\widehat{\mathcal{C}}, \mathcal{D})$.

5.2.5. Proposition. *Let $\mathcal{C}$ be a cofibrant 2-category. For every 2-category $\mathcal{D}$, the canonical inclusion $2\mathcal{C}at(\mathcal{C}, \mathcal{D}) \to 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{D})$ is an equivalence of categories that is natural in $\mathcal{D}$.*

Proof. A quasi-inverse of the canonical inclusion is given by the strictification functor $\widehat{\cdot}$: $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{D}) \to 2\mathcal{C}at(\mathcal{C}, \mathcal{D})$ whose construction is summarised as follows. We refer to [78, §5.2] for the details.

We fix a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$, and define a 2-functor $\widehat{F} : \mathcal{C} \to \mathcal{D}$. On 0-cells, $\widehat{F}$ takes the same values as F. Since $\mathcal{C}$ is cofibrant, its underlying 1-category is free: on generating 1-cells, $\widehat{F}$ is equal to F and, then, it is extended by functoriality on every 1-cell. From the monoidal coherence relations satisfied by F, there is a unique invertible 2-cell $F_a : \widehat{F}(a) \Rightarrow F(a)$, built from the coherence 2-cells of F, for every 1-cell a of $\mathcal{C}$. If $f : a \Rightarrow b$ is a 2-cell of $\mathcal{C}$, we put $\widehat{F}(f) = F_a \star_1 F(b) \star_1 F_b^-$.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be pseudofunctors, and $\alpha : F \Rightarrow G$ be a pseudonatural transformation. Let us define a pseudonatural transformation $\widehat{\alpha} : \widehat{F} \Rightarrow \widehat{G}$. For a 0-cell x of $\mathcal{C}$, we take $\widehat{\alpha}_x = \alpha_x$. If $a : x \to y$ is a 1-cell of $\mathcal{C}$, we define $\widehat{\alpha}_a$ as the following invertible 2-cell of $\mathcal{D}$:

$$\widehat{\alpha}_a \quad = \quad \begin{tikzcd}
 & F(y) \arrow[dr, "\alpha_y"] & \\
F(x) \arrow[ur, "F(a)"'] \arrow[ur, bend left=60, "\widehat{F}(a)"] \arrow[ur, phantom, bend left=30, "F_a"] \arrow[dr, "\alpha_x"'] \arrow[rr, phantom, "\alpha_a"] & & G(y) \\
 & G(x) \arrow[ur, "G(a)"] \arrow[ur, bend right=60, "\widehat{G}(a)"'] \arrow[ur, phantom, bend right=30, "G_a^-"] &
\end{tikzcd}$$

To conclude the proof, it is sufficient to check that, for every pseudofunctor $F : \mathcal{C} \to \mathcal{D}$, there exists a pseudonatural isomorphism $\varphi_F : \widehat{F} \Rightarrow F$ that is itself natural in F. We define φ_F as follows:

(i) If x is a 0-cell of $\mathcal{C}$, then $\widehat{F}(x) = F(x)$ and we take $(\varphi_F)_x = 1_x$.
(ii) If $a : x \to y$ is a 1-cell of $\mathcal{C}$, then we put $(\varphi_F)_a = F_a$.

These data satisfy the required coherence properties, and the naturality condition follows from the definition of $\widehat{\alpha}$. $\square$

5.2.6. Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. The following assertions are equivalent:*

(i) *The 2-categories $\mathcal{C}$ and $\mathcal{D}$ are pseudoequivalent, i.e. there exist pseudofunctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and pseudonatural isomorphisms $GF \simeq 1_{\mathcal{C}}$ and $FG \simeq 1_{\mathcal{D}}$.*
(ii) *For every 2-category $\mathcal{E}$, there exists an equivalence of categories*

$$2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \approx 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E})$$

that is natural in $\mathcal{E}$.

Proof. Assume that $\mathcal{C}$ and $\mathcal{D}$ are pseudoequivalent. As a consequence, for all pseudofunctors $H : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{D} \to \mathcal{E}$, we have $HGF \simeq H$ and $KFG \simeq K$. Thus the functors $2\mathcal{C}at_{\mathrm{ps}}(F, \mathcal{E})$ and $2\mathcal{C}at_{\mathrm{ps}}(G, \mathcal{E})$, respectively mapping pseudofunctors $K : \mathcal{D} \to \mathcal{E}$ to KF and $H : \mathcal{C} \to \mathcal{E}$ to HG, form the required equivalence of categories.

Conversely, assume that, for every 2-category $\mathcal{E}$, we have an equivalence of categories $2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \approx 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E})$ natural in $\mathcal{E}$. This equivalence consists of two functors

$$\Phi_{\mathcal{E}} : 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \to 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E}) \qquad \text{and} \qquad \Psi_{\mathcal{E}} : 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E}) \to 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}),$$

and, for all pseudofunctors $H : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{D} \to \mathcal{E}$, two pseudonatural isomorphisms

$$\Psi_{\mathcal{E}}\Phi_{\mathcal{E}}(H) \simeq H \qquad \text{and} \qquad \Phi_{\mathcal{E}}\Psi_{\mathcal{E}}(K) \simeq K,$$

such that, for all 2-categories $\mathcal{E}$ and $\mathcal{E}'$ and every pseudofunctor $H : \mathcal{E} \to \mathcal{E}'$:

$$\begin{tikzcd}
2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, H)"'] \arrow[dr, phantom, "="] & 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E}) \arrow[r, "\Psi_{\mathcal{E}}"] \arrow[d, "2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, H)" description] \arrow[dr, phantom, "="] & 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \arrow[d, "2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, H)"] \\
2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}') \arrow[r, "\Phi_{\mathcal{E}'}"'] & 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{D}, \mathcal{E}') \arrow[r, "\Psi_{\mathcal{E}'}"'] & 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}').
\end{tikzcd}$$

Define the pseudofunctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ by $F = \Psi_{\mathcal{D}}(1_{\mathcal{D}})$ and $G = \Phi_{\mathcal{C}}(1_{\mathcal{C}})$. The naturality condition on Φ, for $\mathcal{E} = \mathcal{C}$, $\mathcal{E}' = \mathcal{D}$ and $H = F$, implies $F \circ \Phi_{\mathcal{C}}(K) = \Phi_{\mathcal{D}}(F \circ K)$ for every pseudofunctor $K : \mathcal{C} \to \mathcal{C}$. Thus, in the special case $K = 1_{\mathcal{C}}$, we get

$$FG = \Phi_{\mathcal{D}}(F) = \Phi_{\mathcal{D}} \circ \Psi_{\mathcal{D}}(1_{\mathcal{D}}) \simeq 1_{\mathcal{D}}.$$

In a symmetric way, the naturality condition on Ψ gives $GF \simeq 1_{\mathcal{C}}$. □

Proposition 5.2.5 and Lemma 5.2.6 imply

5.2.7. Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, with $\mathcal{D}$ cofibrant. The following assertions are equivalent:*

- ***(i)*** *The 2-category $\mathcal{D}$ is a cofibrant approximation of $\mathcal{C}$.*
- ***(ii)*** *For every 2-category $\mathcal{E}$, there exists an equivalence of categories*

$$2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{E}) \approx 2\mathcal{C}at(\mathcal{D}, \mathcal{E})$$

that is natural in $\mathcal{E}$.

Finally, Theorem 5.2.3 and Proposition 5.2.7 give

5.2.8. Theorem. *Let $\mathcal{C}$ be a category, and X be a $(3,1)$-polygraph that presents $\mathcal{C}$. The following assertions are equivalent:*

- ***(i)*** *X is a coherent presentation of $\mathcal{C}$.*
- ***(ii)*** *For every 2-category $\mathcal{D}$, there is an equivalence of categories*

$$2\mathcal{C}at_{\mathrm{ps}}(\mathcal{C}, \mathcal{D}) \approx 2\mathcal{C}at(\widetilde{X}, \mathcal{D})$$

that is natural in $\mathcal{D}$.

5.3. Tietze equivalence and Tietze transformations

5.3.1. Tietze equivalence of $(3,1)$-polygraphs. Two $(3,1)$-polygraphs X and Y are *Tietze-equivalent* if

- **(i)** the 1-categories $\overline{X}$ and $\overline{Y}$ they present are isomorphic,
- **(ii)** the $(2,1)$-categories $\widetilde{X}$ and $\widetilde{Y}$ they present are equivalent.

As a consequence, two Tietze-equivalent $(3,1)$-polygraphs have the same 0-cells, up to a bijection, and two coherent presentations of the same category are Tietze-equivalent.

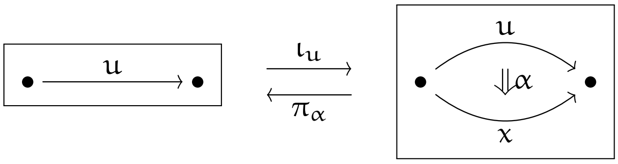
5.3.2. Tietze transformations. Let X be a $(3,1)$-polygraph. A 2-cell (resp. 3-cell, resp. 3-sphere) x of X is *collapsible* if

(i) the target of x is a 1-cell (resp. 2-cell, resp. 3-cell) of X,

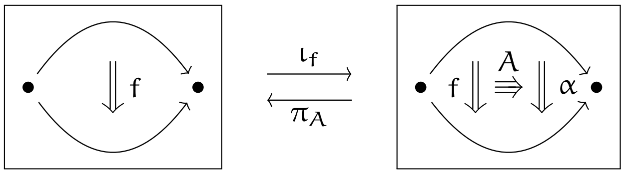
(ii) the source of x satisfies $\mathrm{Cell}(s(x)) \subseteq X \setminus \{t(x)\}$.

An *elementary Tietze transformation of* X is any of the following six operations:

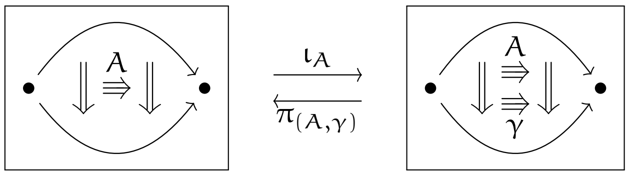
(i) Coherent adjunction or elimination of a 1-cell x and a collapsible 2-cell $\alpha : u \Rightarrow x$:



(ii) Coherent adjunction or elimination of a 2-cell α and a collapsible 3-cell $A : f \Rrightarrow \alpha$ of X:



(iii) Coherent adjunction or elimination of a 3-cell γ:



A set Y of 2-cells (resp. 3-cells, resp. 3-spheres) of X is *collapsible* if, for every x of Y,

(i) the target of x is a 1-cell (resp. 2-cell, resp. 3-cell) of X,

(ii) the source of x satisfies $\mathrm{Cell}(s(x)) \subseteq X \setminus t(Y)$.

The elementary Tietze transformations are generalised in a straightforward way to coherent adjunctions of eliminations of collapsible sets of cells or spheres. If X and Y are $(3,1)$-polygraphs, a *Tietze transformation from* X *to* Y is a composite of generalised elementary Tietze transformations.

5.3.3. Theorem. *Two* $(3,1)$*-polygraphs* X *and* Y *are Tietze-equivalent if, and only if, there exists a Tietze transformation between them.*

Proof. To prove that a Tietze transformation implies Tietze-equivalence, it is sufficient to check the result for each one of the six types of generalised elementary Tietze transformations on a fixed $(3,1)$-polygraph X. By definition, the 3-functors $\pi \circ \iota$ are all equal to the identity of $X^\top$, and the 3-functors $\iota \circ \pi$ induce isomorphisms on the presented categories, and equivalences on the presented 2-categories.

Conversely, assume that X and Y are $(3,1)$-polygraphs, and that $F : \widetilde{X} \to \widetilde{Y}$ is an equivalence. Choose a weak inverse $G : \widetilde{Y} \to \widetilde{X}$ for F, and pseudonatural isomorphisms $\sigma : GF \Rightarrow 1_{\widetilde{X}}$ and $\tau : FG \Rightarrow 1_{\widetilde{Y}}$, in such a way that the quadruple (F, G, σ, τ) is an adjoint equivalence, which is always feasible [154, Chap. IV, § 4, Theorem 1], and means that the relations $F\sigma = \tau F$ and $G\tau = \sigma G$ hold. These data lift to obtain 3-functors $\widehat{F} : X^\top \to Y^\top$ and $\widehat{G} : Y^\top \to X^\top$, 3-cells

$$
\begin{tikzcd}
& GF(u') \arrow[dr, Rightarrow, "\widehat{\sigma}_{u'}"] & \\
GF(u) \arrow[ur, Rightarrow, "\widehat{G}\widehat{F}(\alpha)"] \arrow[dr, Rightarrow, "\widehat{\sigma}_u"'] \arrow[rr, phantom, "\Downarrow\widehat{\sigma}_\alpha"] & & u' \\
& u \arrow[ur, Rightarrow, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& FG(v') \arrow[dr, Rightarrow, "\widehat{\tau}_{v'}"] & \\
FG(v) \arrow[ur, Rightarrow, "\widehat{F}\widehat{G}(\beta)"] \arrow[dr, Rightarrow, "\widehat{\tau}_v"'] \arrow[rr, phantom, "\Downarrow\widehat{\tau}_\beta"] & & v' \\
& v \arrow[ur, Rightarrow, "\beta"'] &
\end{tikzcd}
$$

of $X^\top$ and $Y^\top$, for every 2-cell $\alpha : u \Rightarrow u'$ of X and every 2-cell $\beta : v \Rightarrow v'$ of Y, and 3-cells

$$
\begin{tikzcd}
FGF(x) \arrow[r, Rightarrow, bend left, "\widehat{F}(\widehat{\sigma}_x)"] \arrow[r, Rightarrow, bend right, "\widehat{\tau}_{F(x)}"'] \arrow[r, phantom, "\Downarrow\lambda_x"] & F(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
GFG(y) \arrow[r, Rightarrow, bend left, "\widehat{G}(\widehat{\tau}_y)"] \arrow[r, Rightarrow, bend right, "\widehat{\sigma}_{G(y)}"'] \arrow[r, phantom, "\Downarrow\rho_y"] & G(y)
\end{tikzcd}
$$

of $Y^\top$ and $X^\top$, for every 1-cell x of X and every 1-cell y of Y. Then, we build a $(3, 1)$-polygraph Z that contains both X and Y, together with coherence cells that correspond to the Tietze equivalence. The $(3, 1)$-polygraph Z has the same 0-cells as X (and as Y) and it contains the 1-cells, 2-cells and 3-cells of X and Y, plus the following cells, indexed by cells of X and Y, and extended by functoriality to all cells of $X^\top$ and $Y^\top$:

(i) Two 2-cells $\varphi_x : F(x) \Rightarrow x$ and $\psi_y : G(y) \Rightarrow y$, for all 1-cells x of X and y of Y.

(ii) Two 3-cells φ_α and ψ_β, for all 2-cells $\alpha : u \Rightarrow u'$ of X and $\beta : v \Rightarrow v'$ of Y,

$$
\begin{tikzcd}
& F(u) \arrow[r, Rightarrow, "\widehat{F}(\alpha)"] \arrow[dr, phantom, "\Downarrow\varphi_\alpha"] & F(u') \arrow[dr, Rightarrow, "\varphi_{u'}"] & \\
u \arrow[ur, Rightarrow, "\varphi_u^-"] \arrow[rrr, Rightarrow, "\alpha"'] & & & u'
\end{tikzcd}
\qquad
\begin{tikzcd}
& G(v) \arrow[r, Rightarrow, "\widehat{G}(\beta)"] \arrow[dr, phantom, "\Downarrow\psi_\beta"] & G(v') \arrow[dr, Rightarrow, "\psi_{v'}"] & \\
v \arrow[ur, Rightarrow, "\psi_v^-"] \arrow[rrr, Rightarrow, "\beta"'] & & & v'.
\end{tikzcd}
$$

(iii) Two 3-cells ξ_x and η_y, for all 1-cells x of X and y of Y,

$$
\begin{tikzcd}
& GF(x) \arrow[dr, Rightarrow, "\widehat{\sigma}_x"] \arrow[d, phantom, "\Downarrow\xi_x"] & \\
F(x) \arrow[ur, Rightarrow, "\psi^-_{F(x)}"] \arrow[rr, Rightarrow, "\varphi_x"'] & {} & x
\end{tikzcd}
\qquad
\begin{tikzcd}
& FG(y) \arrow[dr, Rightarrow, "\widehat{\tau}_y"] \arrow[d, phantom, "\Downarrow\eta_y"] & \\
G(y) \arrow[ur, Rightarrow, "\varphi^-_{G(y)}"] \arrow[rr, Rightarrow, "\psi_y"'] & {} & y.
\end{tikzcd}
$$

Finally, we construct a Tietze transformation T from X to Z, step-by-step (see [78, Theorem 2.1.3] for the details): adjunction of the cells y of Y, with all the cells ψ_y; adjunction of the coherence cells φ_x for X, with all the cells ξ_x; adjunction of the remaining coherence cells η_y for Y. By symmetry, we exchange the roles of X and Y, replace coherent adjunctions with coherent eliminations, and reverse the order of the operations to obtain a Tietze transformation U from Z to Y, and consider the composite UT to get the result. □

5.3.4. Corollary. *Assume that $\mathcal{C}$ is a category, and that X and Y are $(3,1)$-polygraphs. If X is a coherent presentation of $\mathcal{C}$, and if there exists a Tietze transformation from X to Y, then Y is a coherent presentation of $\mathcal{C}$.*

Proof. If X is a coherent presentation of $\mathcal{C}$, then the $(2,1)$-category $\widetilde{X}$ is a cofibrant approximation of $\mathcal{C}$ by Theorem 5.2.3. Moreover, if there exists a Tietze transformation from X to Y, then X and Y are Tietze-equivalent by Theorem 5.3.3, so that the $(2,1)$-categories $\widetilde{X}$ and $\widetilde{Y}$ are equivalent. As a consequence, $\widetilde{Y}$ is also a cofibrant approximation of $\mathcal{C}$ and, by Theorem 5.2.3, Y is a coherent presentation of $\mathcal{C}$. □

5.3.5. Higher Nielsen transformations. We introduce higher-dimensional analogues of Nielsen transformations to perform replacements of cells in $(3,1)$-polygraphs. The *elementary Nielsen transformations* on a $(3,1)$-polygraph X are the following operations:

- **(i)** The replacement of a 2-cell or a 3-cell by a formal inverse.
- **(ii)** The replacement of a 3-cell $\gamma : f \Rrightarrow g$ by a 3-cell $\widetilde{\gamma} : h \star_1 f \star_1 k \Rrightarrow h \star_1 g \star_1 k$, where h and k are 2-cells of $X^\top$.

As for elementary Tietze transformations, elementary Nielsen transformations can also be performed on sets of cells. A *Nielsen transformation* is a composition of (generalised) elementary Nielsen transformations. Nielsen transformations are Tietze transformations: indeed, for example, the second one is the composition of the following elementary Tietze transformations, namely the coherent adjunction $\iota_{h\star_1\gamma\star_1 k}$ of a 3-cell $\widetilde{\gamma}$, followed by the coherent elimination $\pi_{(h^-\star_1\widetilde{\gamma}\star_1 k^-,\gamma)}$ of γ.

In what follows, we perform coherent eliminations of cells that are collapsible only up to a Nielsen transformation. If x is equal, up to a Nielsen transformation, to a collapsible cell (or sphere) $\widetilde{x}$, we abusively denote by π_x the corresponding coherent elimination, with a precision about the eliminated cell $t(\widetilde{x})$ when it is not clear from the context.

5.4. Homotopical completion and homotopical reduction

5.4.1. Homotopical completion. Let X be a terminating 2-polygraph, equipped with a total termination order $\leqslant$. A *homotopical completion of* X is a $(3,1)$-polygraph that is a Squier completion of X and whose underlying 2-polygraph is a Knuth-Bendix completion of X with respect to $\leqslant$. Note that, instead of computing a Knuth-Bendix completion and, then, a Squier completion, both constructions can be performed simultaneously by adding the coherence 3-cells during the examination of the critical branchings.

5.4.2. Example. From [122], we consider the following presentation of the braid monoid B_3^+:

$$X = \left(s, t, a \;\middle|\; ta \overset{\alpha}{\Longrightarrow} as\,,\; st \overset{\beta}{\Longrightarrow} a \right)$$

This presentation is obtained from Artin's presentation $(s, t \mid tst \Rightarrow sts)$ by coherent adjunction of the Coxeter element st and the 2-cell β. The deglex order generated by $t > s > a$ proves the termination of X. The following $(3,1)$-polygraph is a homotopical completion of X:

$$\left(s, t, a \;\middle|\; ta \overset{\alpha}{\Longrightarrow} as\,,\; st \overset{\beta}{\Longrightarrow} a\,,\; sas \overset{\gamma}{\Longrightarrow} aa\,,\; saa \overset{\delta}{\Longrightarrow} aat \;\middle|\; A, B, C, D \right)$$

where A, B, C and D are the following 3-cells:

$$
\begin{tikzcd}[column sep=small]
& aa & \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Downarrow A & \\
& sas \arrow[uu, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& aat & \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \Downarrow B & \\
& saa \arrow[uu, Rightarrow, bend right, "\delta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& aaas & \\
sasas \arrow[ur, Rightarrow, "\gamma as"] \arrow[dr, Rightarrow, "sa\gamma"'] & \Downarrow C & aata \arrow[ul, Rightarrow, "aa\alpha"'] \\
& saaa \arrow[ur, Rightarrow, "\delta a"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, "\gamma aa"] \arrow[dr, Rightarrow, "sa\delta"'] & \Downarrow D & \\
& saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}
$$

5.4.3. Homotopical reduction. Let X be a $(3,1)$-polygraph. A *collapsible part of* X is a triple $Y = (Y_2, Y_3, Y_4)$ made of a family Y_2 of 2-cells of X, a family Y_3 of 3-cells of X, and a family Y_4 of 3-spheres of $X^\top$, such that the following conditions are satisfied:

(i) each Y_k is collapsible,

(ii) no y of Y_k appears as the target of an element of Y_{k+1},

(iii) there exists wellfounded order relations on the 1-cells, 2-cells and 3-cells of X such that, for every y in every Y_k, the target of y is strictly greater than every element of $\mathrm{Cell}(s(y))$.

In that case, the recursive assignment

$$
\pi_Y(x) \;=\; \begin{cases} \pi_Y(s(y)) & \text{if } x = t(y) \text{ for } y \text{ in } Y, \\ 1_{\pi_Y(s(x))} & \text{if } x \text{ is in } Y, \\ x & \text{otherwise} \end{cases}
$$

defines, by wellfounded induction, a 3-functor $\pi_Y : X^\top \to X^\top/Y$. The *homotopical reduction of* X *with respect to* Y is the $(3,1)$-polygraph denoted by X/Y and obtained from X by removing the cells of Y and their targets, and by replacing the source and target maps of X by their compositions with π_Y. As a consequence, X/Y is Tietze-equivalent to X, and the free $(3,1)$-category $(X/Y)^\top$ is isomorphic to $X^\top/Y$. In practice, we consider triples $Y = (Y_2, Y_3, Y_4)$ that are only collapsible up to a Nielsen transformation.

Assume that X is a terminating 2-polygraph, with a fixed termination order $\leqslant$. A *homotopical completion-reduction of* X is a $(3,1)$-polygraph obtained by a homotopical reduction with respect to a collapsible part, applied to a homotopical completion of X. Theorems 3.2.6 and 5.3.3 imply

5.4.4. Theorem. *Assume that* X *is a terminating presentation of a category* $\mathcal{C}$. *Then, every homotopical completion-reduction of* X *is a coherent presentation of* $\mathcal{C}$.

5.4.5. Example. In Example 5.4.2, we have obtained a coherent presentation Y of B_3^+ by homotopical completion. We consider the collapsible part Z of Y consisting of the two 3-spheres

$$
\begin{tikzcd}[column sep=small]
& aata \arrow[r, Rightarrow, "aa\alpha"] & aaas \\
& \Downarrow Ba & \\
sasta \arrow[uur, Rightarrow, bend left, "\gamma ta"] \arrow[r, Rightarrow, "sa\beta a"] \arrow[ddr, Rightarrow, bend right, "sas\alpha"'] & saaa \arrow[uu, Rightarrow, bend right, "\delta a"'] & \\
& \Downarrow saA & \\
& sasas \arrow[uu, Rightarrow, bend right, "sa\gamma"'] &
\end{tikzcd}
\quad \overset{\omega_1}{\Rrightarrow} \quad
\begin{tikzcd}[column sep=small]
& aata \arrow[dr, Rightarrow, "aa\alpha"] & & \\
sasta \arrow[ur, Rightarrow, "\gamma ta"] \arrow[dr, Rightarrow, "sas\alpha"'] & = & aaas & \\
& sasas \arrow[ur, Rightarrow, "\gamma as"] \arrow[dr, Rightarrow, "sa\gamma"'] & \Downarrow C & aata \arrow[ul, Rightarrow, "aa\alpha"'] \\
& & saaa \arrow[ur, Rightarrow, "\delta a"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & aaast \arrow[r, Rightarrow, "aaa\beta"] \arrow[d, phantom, "Ct\Downarrow"] & aaaa \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[r, Rightarrow, "sa\gamma t"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] & saaat \arrow[r, Rightarrow, "\delta at"'] \arrow[d, phantom, "\Downarrow saB"] & aatat \arrow[ul, Rightarrow, "aa\alpha t"'] \\
 & sasaa \arrow[u, Rightarrow, bend right, "sa\delta"'] &
\end{tikzcd}
\quad \overset{\omega_2}{\Rrightarrow} \quad
\begin{tikzcd}
 & aaast \arrow[dr, Rightarrow, "aaa\beta"] \arrow[d, phantom, "="] & & \\
sasast \arrow[ur, Rightarrow, "\gamma ast"] \arrow[dr, Rightarrow, "sasa\beta"'] & {} & aaaa \arrow[d, phantom, "\Downarrow D"] & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
 & sasaa \arrow[ur, Rightarrow, "\gamma aa"'] \arrow[dr, Rightarrow, "sa\delta"'] & {} & \\
 & & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}
$$

together with the 3-cells A and B, coherently adjoined with the 2-cells γ and δ during homotopical completion, and the 2-cell $\beta : st \Rightarrow a$ that defines the redundant generator a. We have that ω_1, ω_2, A, B and β are collapsible (up to a Nielsen transformation), with respective targets C, D, γ, δ and a. We conclude that Z is collapsible with the wellfounded orders

$$D > C > B > A, \qquad \delta > \gamma > \beta > \alpha, \qquad a > t > s.$$

Thus the homotopical reduction Y/Z is the $(3,1)$-polygraph $(s, t \,|\, tst \Rightarrow sts \,|\, \emptyset)$ which, by Theorem 5.4.4, is a coherent presentation of B_3^+.

5.5. Garside's coherent presentation of Artin monoids

5.5.1. Recollections on Coxeter groups. Assume that W is a Coxeter group. For $u \in W$, a *reduced expression of* u is a representative of minimal length of u in S^*, and the *length of* u is denoted by $l(u)$ and defined as the length of any of its reduced expressions. The Coxeter group W is finite if, and only if, it admits an element of maximal length [36, Theorem 5.6]; in that case, this element is unique and denoted by $w_0(S)$. For $I \subseteq S$, the subgroup of W spanned by I is also a Coxeter group, denoted by W_I.

For all u and v in W, we have $l(uv) \leqslant l(u) + l(v)$ and we use distinct graphical notations depending on whether the equality holds or not:

$$u \frown v \Leftrightarrow l(uv) = l(u) + l(v) \qquad \text{and} \qquad u^{\times} v \Leftrightarrow l(uv) < l(u) + l(v).$$

We generalise the notation for a greater number of elements of W. For example, in the case of three elements u, v and w of W, we write $u \frown v \frown w$ if both equalities $l(uv) = l(u) + l(v)$ and $l(vw) = l(v) + l(w)$ hold. This situation splits in the following two mutually exclusive subcases:

$$
\begin{aligned}
\overset{\frown}{u \frown v \frown w} \quad &\Leftrightarrow \quad \Big(u \frown v \frown w \quad \text{and} \quad l(uvw) = l(u) + l(v) + l(w) \Big), \\
u \frown \overset{\times}{v} \frown w \quad &\Leftrightarrow \quad \Big(u \frown v \frown w \quad \text{and} \quad l(uvw) < l(u) + l(v) + l(w) \Big).
\end{aligned}
$$

5.5.2. Artin monoids. Fix a Coxeter group W. The *Artin monoid associated to* W is the monoid denoted by $B^+(W)$, with the same presentation as W except for the relations $s^2 = 1$. If u and v are elements of $B^+(W)$, we say that u *is a divisor of* v, or that v *is a multiple of* u, if there exists an element u' in $B^+(W)$ such that $uu' = v$. In that case, the element u' is unique and called the *complement of* u *in* v [36, Proposition 2.3]; moreover, if v is in W, seen as an element of $B^+(W)$

by the canonical embedding (given by Matsumoto's theorem, see [80, Theorem 1.2.2]), then we also have u and u' in W and $u \frown u'$. If two elements u and v of $B^+(W)$ have a common multiple, then they have a least common multiple, lcm for short [36, Proposition 4.1].

The Artin monoid $B^+(W)$ admits, as an alternative presentation, the 2-polygraph $\mathrm{Gar}_2(W)$, called *Garside's presentation of* $B^+(W)$, whose 1-cells are the elements of $W_+ = W \setminus \{1\}$, and with one 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$ whenever $u \frown v$ holds. Here, we write uv for the product in W and $u|v$ for the product in the free monoid over W.

5.5.3. Proposition. *For every Coxeter group W, the monoid $B^+(W)$ admits, as a coherent presentation, the $(3,1)$-polygraph with one 0-cell, one 1-cell for every element of W_+, the 2-cells*

$$u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv \qquad \textit{and} \qquad u|vw \overset{\beta_{u,v,w}}{\Longrightarrow} uv|w,$$

respectively for all u, v of W_+ with $u \frown v$, and all u, v, w of W_+ with $u \frown \overset{\times}{v} \frown w$, and the following nine families of 3-cells, indexed by all the possible elements of W_+:

$$\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "\Downarrow A_{u,v,w}"] & & uvw \\
& u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "\Downarrow B_{u,v,w}"] & & uv|w \\
& u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|wx \arrow[dr, Rightarrow, "\beta_{uv,w,x}"] & \\
u|v|wx \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] \arrow[rr, phantom, "\Downarrow C_{u,v,w,x}"] & & uvw|x \\
& u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
u|v|wx \arrow[rrr, Rightarrow, bend left, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] \arrow[rrr, phantom, "\Downarrow D_{u,v,w,x}"] & & & uv|wx \\
& u|vw|x \arrow[r, Rightarrow, "\beta_{u,v,w}|x"'] & uv|w|x \arrow[ur, Rightarrow, "uv|\alpha_{w,x}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|w|x \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"] & \\
u|vw|x \arrow[ur, Rightarrow, "\beta_{u,v,w}|x"] \arrow[dr, Rightarrow, "u|\alpha_{vw,x}"'] \arrow[rr, phantom, "\Downarrow E_{u,v,w,x}"] & & uv|wx \\
& u|vwx \arrow[ur, Rightarrow, "\beta_{u,v,wx}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& uv|w|xy \arrow[rr, Rightarrow, "uv|\alpha_{w,xy}"] & & uv|wxy \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] \arrow[rrr, phantom, "\Downarrow F_{u,v,w,x,y}"] & & & \\
& u|vwx|y \arrow[rr, Rightarrow, "\beta_{u,v,wx}|y"'] & & uv|wx|y \arrow[uu, Rightarrow, "uv|\alpha_{wx,y}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|w|xy \arrow[dr, Rightarrow, "uv|\beta_{w,x,y}"] & \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] \arrow[rr, phantom, "\Downarrow G_{u,v,w,x,y}"] & & uv|wx|y \\
& u|vwx|y \arrow[ur, Rightarrow, "\beta_{u,v,wx}|y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& uv|xy \arrow[dr, Rightarrow, "\beta_{uv,x,y}"] & \\
u|vxy \arrow[ur, Rightarrow, "\beta_{u,v,xy}"] \arrow[rr, Rightarrow, bend right, "\beta_{u,vx,y}"'] \arrow[rr, phantom, "\Downarrow H_{u,v,x,y}"] & & uvx|y
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv_1|w_1 = uv_1|x_1y \arrow[dr, Rightarrow, "\beta_{uv_1,x_1,y}"] & \\
u|v_1w_1 = u|v_2w_2 \arrow[ur, Rightarrow, "\beta_{u,v_1,w_1}"] \arrow[dr, Rightarrow, "\beta_{u,v_2,w_2}"'] \arrow[rr, phantom, "\Downarrow I_{u,v_1,w_1,v_2,w_2}"] & & uv_1x_1|y = uv_2x_2|y \\
& uv_2|w_2 = uv_2|x_2y \arrow[ur, Rightarrow, "\beta_{uv_2,x_2,y}"'] &
\end{tikzcd}$$

Proof. Let $\leqslant$ denote the strict order on the elements of the free monoid W^* that first compares their length as elements of W^*, and then the length of their components as elements of W, starting from the right. For example, $u_1|u_2 < v_1|v_2|v_3$ (first condition) and $uv|w < u|vw$ if $u^\frown v^\frown w$ (second condition). This is a termination order on $\mathrm{Gar}_2(W)$: for every 2-cell $\alpha_{u,v}$ of $\mathrm{Gar}_2(W)$, we have $u|v > uv$. Hence the 2-polygraph $\mathrm{Gar}_2(W)$ terminates. Then, we check that the $(3,1)$-polygraph of Proposition 5.5.3 is a homotopical completion of $\mathrm{Gar}_2(W)$ with respect to $\leqslant$. The 2-polygraph $\mathrm{Gar}_2(W)$ has exactly one critical branching for all u, v and w of W_+ such that $u^\frown v^\frown w$:

$$uv|w \xLeftarrow{\alpha_{u,v}|w} u|v|w \xRightarrow{u|\alpha_{v,w}} u|vw$$

Then there are two possibilities: if $u^\frown \widehat{v}{}^\frown w$, the branching is confluent (generating the 3-cell $A_{u,v,w}$), and, if $u^\frown \overset{\times}{v}{}^\frown w$, the branching is made confluent by adjunction of the 2-cell $\beta_{u,v,w}$ (generating the 3-cell $B_{u,v,w}$). The family β of 2-cells creates new critical branchings, each one being confluent, generating all the other 3-cells $C, \ldots, I$, see [78, Proposition 3.2.1]. $\square$

5.5.4. Garside's coherent presentation. Let W be a Coxeter group. We call *Garside's coherent presentation of* $\mathrm{B}^+(W)$, and denote by $\mathrm{Gar}_3(W)$, the $(3,1)$-polygraph obtained from $\mathrm{Gar}_2(W)$ by adjunction of one 3-cell

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "\Downarrow A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for all u, v and w of W_+ such that $u^\frown \widehat{v}{}^\frown w$.

5.5.5. Theorem. *For every Coxeter group W, the Artin monoid $\mathrm{B}^+(W)$ admits $\mathrm{Gar}_3(W)$ as a coherent presentation.*

Proof. We use homotopical reduction on the $(3,1)$-polygraph of Proposition 5.5.3 with the following collapsible part: seven families of 3-spheres generated by the triple critical branchings, with targets the 3-cells $C, \ldots, I$, and the family B of 3-cells, with target the 2-cells β, see [78, §3.2.2]. The result is $\mathrm{Gar}_3(W)$, and we invoke Theorem 5.4.4 to conclude. $\square$

Theorems 5.2.8 and 5.5.5 imply

5.5.6. Corollary ([67, Theorem 1.5]). *Let W be a Coxeter group. Then, for every 2-category $\mathcal{C}$, the categories $2\mathcal{C}at_{\mathrm{ps}}(\mathrm{B}^+(W), \mathcal{C})$ and $2\mathcal{C}at(\widetilde{\mathrm{Gar}_3(W)}, \mathcal{C})$ are equivalent, and this equivalence is natural in $\mathcal{C}$.*

5.6. Artin's coherent presentation of Artin monoids

5.6.1. Artin's coherent presentation. Fix a Coxeter group W, with a totally ordered set S of generators. We call *Artin's presentation of* $\mathrm{B}^+(W)$ the 2-polygraph $\mathrm{Art}_2(W)$ with one 0-cell, the elements of S as 1-cells, and one 2-cell $\gamma_{s,t} : \langle ts\rangle^{m_{st}} \Rightarrow \langle st\rangle^{m_{st}}$ for each braid relation and for $t > s$.

5.6.2. Classification of the cells of Garside's coherent presentation. If u is an element of W_+, the *smallest divisor of* u is denoted by d_u and defined as the smallest element of S that is a divisor of u (in W). Let $(u_1, \ldots, u_n)$ be a family of elements of W_+ such that

$$l(u_1 \cdots u_n) \;=\; l(u_1) + \cdots + l(u_n).$$

For every $k \in \{1, \ldots, n\}$, we write $s_k = d_{u_1 \cdots u_k}$. We have $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$ since each s_k divides $u_1 \cdots u_l$ for $l \geqslant k$. Moreover, the elements $s_1, \ldots, s_k$ have $u_1 \cdots u_k$ as common multiple, so that their lcm $w_0(s_1, \ldots, s_k)$ exists and divides $u_1 \cdots u_k$, and each subgroup $W_{\{s_1,\ldots,s_k\}}$ is finite.

We say that $(u_1, \ldots, u_n)$ is *essential* if the following conditions are satisfied:

(i) $u_1 \in S$,

(ii) u_{k+1} is the complement of $w_0(s_1, \ldots, s_k)$ in $w_0(s_1, \ldots, s_{k+1})$ for every $k < n$.

In that case, $u_1 = s_1$ and we have $s_1 > \cdots > s_n$, since each u_k is different from 1. Thus, $(u_1, \ldots, u_n)$ is uniquely determined by the elements $s_1, \ldots, s_n$ of S such that $s_1 > \cdots > s_n$.

Assume that $(u_1, \ldots, u_n)$ is not essential, and set k as the minimal element of $\{1, \ldots, n\}$ such that $(u_1, \ldots, u_k)$ is not essential. If $k \geqslant 2$, there are two possibilities, depending if $w_0(s_1, \ldots, s_{k-1}) = w_0(s_1, \ldots, s_k)$ or not, which is equivalent to the equality $s_{k-1} = s_k$ since $s_1 > \cdots > s_{k-1} \geqslant s_k$. If $s_{k-1} = s_k$, we say that $(u_1, \ldots, u_n)$ is *collapsible*. If $s_{k-1} > s_k$, then we have $u_k = vw$ with v and w in W_+ such that $v \frown w$ and $(u_1, \ldots, u_{k-1}, v)$ is essential: we say that $(u_1, \ldots, u_n)$ is *redundant*. We also say that $(u_1, \ldots, u_n)$ is *redundant* if $k = 1$, in which case $u_1 = s_1 w$ with w in W_+.

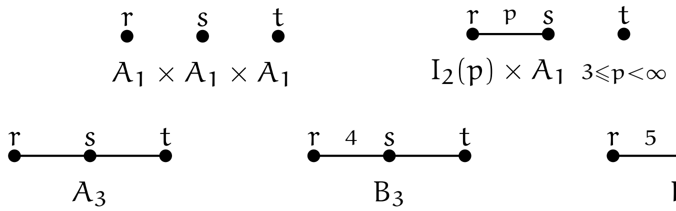
By definition, the family $(u_1, \ldots, u_n)$ is either essential, collapsible or redundant, giving a three-block partition of the cells of $\mathrm{Gar}_3(W)$ and 3-spheres of $\mathrm{Gar}_3(W)^\top$.

5.6.3. From Garside's to Artin's coherent presentation. We define X as the collection of all the 2-cells and 3-cells of $\mathrm{Gar}_3(W)$ and the 3-spheres of $\mathrm{Gar}_3(W)^\top$ whose indexing family is collapsible. This is a collapsible part of $\mathrm{Gar}_3(W)$, see the details in [78, §4.2]. For example, the 3-cells of X are the

$$\begin{tikzcd}[column sep=small] & su|v \arrow[dr, Rightarrow, "\alpha_{su,v}"] & \\ s|u|v \arrow[ur, Rightarrow, "\alpha_{s,u}|v"] \arrow[dr, Rightarrow, "s|\alpha_{u,v}"'] \arrow[rr, phantom, "\Downarrow A_{s,u,v}"] & & suv \\ & s|uv \arrow[ur, Rightarrow, "\alpha_{s,uv}"'] & \end{tikzcd}$$

with either (a) $s = d_{su}$ or (b) $s > d_{su} = d_{suv}$ and $su = w_0(s, d_{su})$. Those 3-cells are collapsible up to a Nielsen transformation, with target the 2-cells: (a) $\alpha_{su,v}$ or (b) $\alpha_{s,uv}$. By hypothesis, the corresponding indexing pair (su, v) or (s, uv) is redundant, so that none of those 2-cells is in X. Finally, we use a wellfounded order on 2-cells that compares, alternatively, the lengths of indices and their smallest divisor in S, to check that each redundant 2-cell is strictly greater than the other 2-cells appearing in the source and target of $A_{s,u,v}$.

As a consequence, the homotopical reduction of $\mathrm{Gar}_3(W)$ with respect to X eliminates all the cells, except for the ones whose indexing family is essential, and the remaining cells have their source and target modified by application of the projection π_X. Thus, the 1-cells of $\mathrm{Gar}_3(W)/X$ are the elements of S. The essential 2-cells of $\mathrm{Gar}_3(W)/X$ are the $\alpha_{s,u}$ such that $s > d_{su}$ and $su = w_0(s, d_{su})$: there is one such 2-cell for all $t > s$ in s such that $W_{\{s,t\}}$ is finite, with source $\langle ts\rangle^{m_{st}}$ and target $\langle st\rangle^{m_{st}}$. Thus, the 2-polygraph underlying $\mathrm{Gar}_3(W)/X$ is isomorphic to Artin's presentation $\mathrm{Art}_2(W)$ of $B^+(W)$. Finally, the essential 3-cells are the $A_{s,u,v}$ such that $s > d_{su} > d_{suv}$, $su = w_0(s, d_{su})$ and $suv = w_0(s, d_{su}, d_{suv})$. Hence, there is one such 3-cell for all $t > s > r$ in S such that $W_{\{r,s,t\}}$ is finite. According to the classification of finite Coxeter groups [33, Chap. VI, §4, Theorem 1], there are five types of finite Coxeter groups of rank 3:



For each case, the inductive definition of the projection π_X is used to compute the shape of each 3-cell $Z_{r,s,t} = \pi_X(A_{s,u,v})$, for each one of the five cases. In practice, we have written a small Python library for these computations [94], based on the PyCox library [79]. The results are given in the following theorem, and can also be found in string diagrams in [70, Definition 4.3].

5.6.4. Theorem. *Fix a Coxeter group* W*, with a totally ordered set of generators* S*. The Artin monoid* $B^+(W)$ *admits, as a coherent presentation, the* $(3,1)$*-polygraph* $\mathrm{Art}_3(W)$ *made of Artin's presentation* $\mathrm{Art}_2(W)$*, extended with one* 3*-cell* $Z_{r,s,t}$ *for all elements* $t > s > r$ *of* S *such that the subgroup* $W_{\{r,s,t\}}$ *is finite, and whose shape depends only on the Coxeter type of* $W_{\{r,s,t\}}$*:*

$$
\begin{tikzcd}
 & str \arrow[r, Rightarrow, "s\gamma_{rt}"] & srt \arrow[dr, Rightarrow, "\gamma_{rs}t"] & \\
tsr \arrow[ur, Rightarrow, "\gamma_{st}r"] \arrow[dr, Rightarrow, "t\gamma_{rs}"'] & & Z_{r,s,t}(A_1^3) & rst \\
 & trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & rts \arrow[ur, Rightarrow, "r\gamma_{st}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & st\langle rs\rangle^{p-1} \arrow[r, Rightarrow, "s\gamma_{rt}\langle rs\rangle^{p-2}"] & (\cdots) \arrow[r, Rightarrow] & \langle sr\rangle^{p}t \arrow[dr, Rightarrow, "\gamma_{rs}t"] & \\
t\langle sr\rangle^{p} \arrow[ur, Rightarrow, "\gamma_{st}\langle rs\rangle^{p-1}"] \arrow[dr, Rightarrow, "t\gamma_{rs}"'] & & Z_{r,s,t}(I_2(p)\times A_1) & & \langle rs\rangle^{p}t \\
 & t\langle rs\rangle^{p} \arrow[r, Rightarrow, "\gamma_{rt}\langle sr\rangle^{p-1}"'] & rt\langle sr\rangle^{p-1} \arrow[r, Rightarrow, "r\gamma_{st}\langle sr\rangle^{p-2}"'] & (\cdots) \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & strsrt \arrow[r, Rightarrow, "s\gamma_{rt}s\gamma_{rt}^{-}"] & srtstr \arrow[r, Rightarrow, "sr\gamma_{st}r"] & srstsr \arrow[dr, Rightarrow, "\gamma_{rs}tsr"] & \\
stsrst \arrow[ur, Rightarrow, "st\gamma_{rs}t"] & & & & rsrtsr \\
tstrst \arrow[u, Rightarrow, "\gamma_{st}rst"] \arrow[d, Rightarrow, "ts\gamma_{rt}st"'] & & Z_{r,s,t}(A_3) & & rstrsr \arrow[u, Rightarrow, "rs\gamma_{rt}sr"'] \\
tsrtst \arrow[dr, Rightarrow, "tsr\gamma_{st}"'] & & & & rstsrs \arrow[u, Rightarrow, "rst\gamma_{rs}"'] \\
 & tsrsts \arrow[r, Rightarrow, "t\gamma_{rs}ts"'] & trsrts \arrow[r, Rightarrow, "\gamma_{rt}s\gamma_{rt}^{-}s"'] & rtstrs \arrow[ur, Rightarrow, "r\gamma_{st}rs"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& srtsrtstr \arrow[r, Rightarrow, "srts\gamma^-_{rt}str"] & srtstrstr \arrow[r, Rightarrow, "sr\gamma_{st}rs\gamma_{rt}"] & srstsrsrt \arrow[r, Rightarrow, "srst\gamma_{rs}t"] & srstrsrst \arrow[r, Rightarrow, "srs\gamma_{rt}srst"] & srsrtsrst \arrow[dr, Rightarrow, bend left, "\gamma_{rs}tsrst"] & \\
strsrstsr \arrow[ur, Rightarrow, bend left, "s\gamma_{rt}sr\gamma^-_{st}r"] & & & & & & rsrstsrst \\
stsrsrtsr \arrow[u, Rightarrow, "st\gamma_{rs}tsr"'] & & & & & & rsrtstrst \arrow[u, Rightarrow, "rsr\gamma_{st}rst"] \\
tstrsrtsr \arrow[u, Rightarrow, "\gamma_{st}rsrtsr"'] \arrow[d, Rightarrow, "ts\gamma_{rt}s\gamma^-_{rt}sr"'] & & & Z_{r,s,t}(B_3) & & & rsrtsrtst \arrow[u, Rightarrow, "rsrts\gamma^-_{rt}st"] \\
tsrtstrsr \arrow[d, Rightarrow, "tsr\gamma_{st}rsr"'] & & & & & & rstrsrsts \arrow[u, Rightarrow, "rs\gamma_{rt}sr\gamma^-_{st}"] \\
tsrstsrsr \arrow[dr, Rightarrow, bend right, "tsrst\gamma_{rs}"'] & & & & & & rstsrsrts \arrow[u, Rightarrow, "rst\gamma_{rs}ts"] \\
& tsrstrsrs \arrow[r, Rightarrow, "tsrs\gamma_{rt}srs"'] & tsrsrtsrs \arrow[r, Rightarrow, "t\gamma_{rs}tsrs"'] & trsrstsrs \arrow[r, Rightarrow, "\gamma_{rt}sr\gamma^-_{st}rs"'] & rtsrtstrs \arrow[r, Rightarrow, "rts\gamma^-_{rt}strs"'] & rtstrstrs \arrow[ur, Rightarrow, bend right, "r\gamma_{st}rs\gamma_{rt}s"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
& srstrsrsrtsrsrt \arrow[r, Rightarrow] & srsrtsrstrsrsrt \arrow[r, Rightarrow] & srsrtsrstsrsrst \arrow[r, Rightarrow] & srsrtsrtstrsrst \arrow[dr, Rightarrow, bend left] & \\
srstsrsrstsrsrt \arrow[ur, Rightarrow, bend left] & & & & & srsrtstrsrtsrst \arrow[d, Rightarrow] \\
srtstrsrtstrsrt \arrow[u, Rightarrow] & & & & & srsrstsrsrtsrst \arrow[d, Rightarrow] \\
srtsrtstrsrtstr \arrow[u, Rightarrow] & & & & & rsrsrtsrsrtsrst \\
srtsrstsrsrstsr \arrow[u, Rightarrow] & & & & & rsrstrsrsrtsrst \arrow[u, Rightarrow] \\
srtsrstrsrsrtsr \arrow[u, Rightarrow] & & & & & rsrstsrsrstsrst \arrow[u, Rightarrow] \\
strsrsrtsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtstrsrtstrst \arrow[u, Rightarrow] \\
stsrsrstsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtsrtstrsrtst \arrow[u, Rightarrow] \\
tstrsrstsrsrtsr \arrow[u, Rightarrow] \arrow[d, Rightarrow] & & Z_{r,s,t}(H_3) & & & rsrtsrstsrsrsts \arrow[u, Rightarrow] \\
tsrtsrstsrstrsr \arrow[d, Rightarrow] & & & & & rsrtsrstrsrsrts \arrow[u, Rightarrow] \\
tsrtsrtstrstrsr \arrow[d, Rightarrow] & & & & & rstrsrsrtsrsrts \arrow[u, Rightarrow] \\
tsrtstrsrtstrsr \arrow[d, Rightarrow] & & & & & rstsrsrstsrsrts \arrow[u, Rightarrow] \\
tsrstsrsrstsrsr \arrow[d, Rightarrow] & & & & & rtstrsrtstrsrts \arrow[u, Rightarrow] \\
tsrstrsrsrtsrsr \arrow[d, Rightarrow] & & & & & rtsrtstrsrtstrs \arrow[u, Rightarrow] \\
tsrsrtsrstrsrsr \arrow[d, Rightarrow] & & & & & rtsrstsrsrstsrs \arrow[u, Rightarrow] \\
tsrsrtsrstsrsrs \arrow[dr, Rightarrow, bend right] & & & & & rtsrstrsrsrtsrs \arrow[u, Rightarrow] \\
& tsrsrtsrtstrsrs \arrow[r, Rightarrow] & tsrsrtstrsrtsrs \arrow[r, Rightarrow] & tsrsrstsrsrtsrs \arrow[r, Rightarrow] & trsrsrtsrsrtsrs \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}
$$

From Theorem 5.6.4, we deduce

5.6.5. Corollary ([201, Proposition 4], [186, Theorem 2.17]). *Fix a Coxeter group W, with generating set S. The monoid $B^+(W)$ admits, as a coherent presentation, Artin's presentation extended with all the 3-spheres of $\mathrm{Art}_2(W)^\top$ whose 1-source is a reduced expression of W_I, where I ranges among the three-element subsets of S such that W_I is finite.*

Theorems 5.2.8 and 5.6.4 imply the following generalisation of [67, §1.3]:

5.6.6. Corollary. *Let W be a Coxeter group W. Then, for every 2-category $\mathcal{C}$, the categories $2\mathcal{C}at_{\mathrm{ps}}(B^+(W), \mathcal{C})$ and $2\mathcal{C}at(\widetilde{\mathrm{Art}_3}, \mathcal{C})$ are equivalent, and this equivalence is natural in $\mathcal{C}$.*

Chapter 6
Quadratic normalisations for monoids

In this chapter, given a set S and a natural number p, we identify S^p with the length-p elements of the free monoid S^*. If $\varphi : S^p \to S^p$ is a map, for $n \geqslant p$ and $1 \leqslant i \leqslant n-p+1$, we denote by $\varphi_i : S^n \to S^n$ the map that applies φ to the entries in positions $i \ldots i+p-1$, and, for $i_1, \ldots, i_n \geqslant 1$, we write $\varphi_{i_1 \cdots i_n}$ for the composite $\varphi_{i_n} \cdots \varphi_{i_1}$. If $\varphi : S^* \to S^*$ is a graded map, we write $\overline{\varphi}$ for its restriction to S^2, and $\varphi^{(p)}$ for its restriction to S^p. Finally, for $m \geqslant 0$, we write $12[m]$ and $21[m]$ for the alternating elements 121... and 212... of $\{1,2\}^*$ of length m.

6.1. Introduction

6.1.1. Context. A normal form for a monoid M, with a specified generating subfamily S, is a map that assigns to each element of M a distinguished representative element of S^*. Our aim in [63] was to investigate a certain type of such normal forms and, more precisely, the associated normalisation processes, that is, the syntactic transformations that lead from an arbitrary element of S^* to a normal one. There, we restricted to geodesic normal forms, which select representatives of minimal length, and investigated the quadratic case, that is, when some locality conditions are satisfied: that an element of S^* is normal if, and only if, each of its length-two factors are normal, and that one can always transform an element of S^* into a normal one by a finite sequence of steps, each of which normalising a length-two factor.

This general framework includes two well-known classes of normalisation processes: those associated with Garside families as investigated in [62] and [60], building on the seminal example of the greedy normal form in Artin monoids [1, 185, 71], and those associated with quadratic 2-polygraphs as investigated for instance in [78] for Artin monoids and in [29, 44] for plactic monoids. So the present development can be seen as an effort to unify various approaches and understand their common features. This programme is made natural by the observation that, in spite of their unrelated definitions, the normalisation processes arising in the above mentioned situations share common mechanisms: for instance, in each case, an element of S^3 can be normalised in three steps, successively normalising the length-two factors in position 2-3, then in position 1-2, and in position 2-3 again.

Krammer's ideas had a seminal influence in our approach, in particular for the connection between normalisation and the monoid underlying normalisations of class $(4,3)$, which he investigated in [130]. A similar connection was independently discovered by Hess and Ozornova in [107, 172, 108], partly building on unpublished work by Rodenhausen. The approach presented in this chapter is close to theirs in the case of graded monoids. In this case, beyond minor terminology discrepancies, the factorability structures of [108] correspond to what we call normalisations of class $(4, 3)$. But, in the general case, the two viewpoints are not directly comparable because of divergent treatment of units and invertible elements: in both settings a "dummy" element is used, but with different assumptions, resulting in different notions of complexity and different conclusions. It seems that every factorability structure yields a normalisation of class $(4, 5)$, but understanding which normalisations of class $(4, 5)$ arise in this way remains open.

6.1.2. Summary. The central technical notion is that of a *normalisation*, introduced in §6.2, which is a pair (S, N) made of a set S and an idempotent length-preserving map N from the free monoid S^* to itself: the intuition is that $N(w)$ is the result of normalising w, that is, $N(w)$ is the distinguished element in the equivalence class of w. The normalisation automatically determines the associated monoid via the defining relations $w = N(w)$, and we take it as our basic object of investigation. We call *quadratic* a normalisation (S, N) such that an element w of S^* is N-normal (meaning $N(w) = w$) if, and only if, each length-two factor of w is N-normal, and such that one can go from w to $N(w)$ by applying a finite sequence of shifted copies of the restriction $\overline{N}$ of N to S^2.

We then introduce, in §6.3, for every quadratic normalisation, a *class*, which is a pair of natural numbers describing the complexity of normalisation on S^3: by definition, if w belongs to S^3, then $N(w)$ is equal to $\overline{N}_{21[m]}(w)$ or $\overline{N}_{12[m]}(w)$, meaning a length-m sequence of alternate applications of $\overline{N}$ in positions 1-2 and 2-3, and we say that the class is (m, n) if one always reaches the normal form after at most m steps when starting from the left, and n steps from the right. We observe that the class, if not infinite, has the form (m, n) with $|m - n| \leqslant 1$, and that a quadratic normalisation of class (m, n) is of class (m', n') for all $m' \geqslant m$ and $n' \geqslant n$. We give a number of examples witnessing possible behaviours for the class and its analogue for the normalisation of longer elements of S^* in §3.3 of [63]. However, most of our general results involve quadratic normalisations of class $(4, 3)$ or $(3, 4)$.

The first main result, recalled in §6.4, is an axiomatisation of normalisations of class $(4, 3)$ in terms of the restriction of the normalisation map to S^2:

Theorem 6.4.7. *If* (S, N) *is a quadratic normalisation of class* $(4, 3)$*, then the restriction* $\overline{N}$ *of* N *to* S^2 *is idempotent and satisfies* $\overline{N}_{212} = \overline{N}_{2121} = \overline{N}_{1212}$*. Conversely, if* φ *is an idempotent map on* S^2 *that satisfies* $\varphi_{212} = \varphi_{2121} = \varphi_{1212}$*, there exists a quadratic normalisation* (S, N) *of class* $(4, 3)$ *satisfying* $\varphi = \overline{N}$.

The direct implication extends to quadratic normalisations of classes higher than $(4, 3)$, but the converse direction does not: a map on S^2 normalising elements of S^3 needs not normalise elements of S^p for $p > 3$. The proof of Theorem 6.4.7 involves the monoid M_p studied in [130] and [108], which is an asymmetric version of Artin monoids where the braid relation $s_2s_1s_2 = s_1s_2s_1$ is replaced with $s_2s_1s_2 = s_1s_2s_1s_2 = s_2s_1s_2s_1$. Let us mention that [108, Theorem 3.4] is an analogue of Theorem 6.4.7 for factorability structures.

The second main result, in §6.5, involves termination. Every quadratic normalisation (S, N) gives rise to a quadratic 2-polygraph, namely the one with 2-cells $w \Rightarrow \overline{N}(w)$ for every non-N-normal element w of S^2. By construction, this 2-polygraph is confluent and normalising, meaning that, starting from any element of S^*, there exists a finite sequence of rewriting steps leading to a unique N-normal element of S^*, but its convergence, meaning also that *any* sequence of rewriting steps is finite, is a different question. We prove

Theorem 6.5.3. *If* (S, N) *is a quadratic normalisation of class* $(3, 4)$ *or* $(4, 3)$*, then the associated 2-polygraph is convergent, with every sequence of rewriting steps starting from an element of* S^p *having length at most* $2^p - p - 1$.

The result can be compared with Proposition 6.5.1, stating that, in the class $(3, 3)$, every sequence of rewriting steps starting from an element of S^p has length at most $p(p-1)/2$, and it is optimal, in the sense that there exists a quadratic normalisation of class $(4, 4)$ whose associated 2-polygraph is not convergent. The proof of Theorem 6.5.3 is delicate and relies on a diagrammatic tool called the domino rule. Theorem 6.5.3 exhibits a strong difference between the factorability structures of [108] and normalisations of class $(4, 3)$, since the former can induce nonterminating 2-polygraphs, as witnessed by the counterexample of [108, Appendix, Proposition 7]. However, there is a connection between Theorem 6.5.3 and [108, Theorem 7.3], which states termination in the case of a factorability structure that obeys the domino rule, hence, as a normalisation, is of class $(4, 3)$. The arguments are different, and it is not clear how restrictive it is for a normalisation of class $(4, 3)$ to be associated with a factorability structure.

As mentioned above, Garside normalisation [60] integrates into quadratic normalisations, more precisely normalisations of class $(3, 3)$ in the case of a bounded Garside family, and of class $(4, 3)$ in the general case. It is natural to ask for a characterisation of Garside systems inside the family of all normalisations of class $(4, 3)$. This is the last one of our main results, proved in §6.6:

Theorem 6.6.3. *Call a normalisation* (S, N) *left-weighted if, for all* s *and* t *in* S*, the element* s *left-divides the first entry of* $N(st)$ *in the associated monoid. Then, for every normalisation* (S, N) *such that the associated monoid* M *is left-cancellative and contains no nontrivial invertible element, the family* S *is a Garside family in* M *and* (S, N) *is the derived normalisation if, and only if,* (S, N) *is of class* $(4, 3)$ *and left-weighted.*

The proof relies on nontrivial properties of Garside families and, again, on the domino rule available in class $(4, 3)$. A consequence of Theorems 6.5.3 and 6.6.3 is that the 2-polygraph derived from a Garside family is always convergent, which generalises the case of Artin monoids with the elements of the corresponding Coxeter group as generators [78, Theorem 3.1.3, Proposition 3.2.1].

Note that almost all observations in this chapter extend from the context of monoids to that of categories, seen as monoids with a partially defined product.

6.2. Quadratic normalisations

6.2.1. Normalisations. A *normalisation* is a pair (S, N), where S is a set and $N : S^* \to S^*$ is a map satisfying, for all u, v and w in S^*,

(i) $\|N(u)\| = \|u\|$,
(ii) $\|u\| = 1 \Rightarrow N(u) = u$,
(iii) $N(uN(v)w) = N(uvw)$.

Note that the last condition implies that N is idempotent. An element u of S^* satisfying $N(u) = u$ is called N-*normal*. If M is a monoid, we say that (S, N) is a normalisation *for* M if

$$M \simeq \langle S \,|\, (N(u) = u)_{u \in S^*} \rangle^+ .$$

Assume that M is a monoid, S is a generating subset of M, and $\pi : S^* \twoheadrightarrow M$ is the canonical projection. If $N : S^* \to S^*$ is a length-preserving map, then (S, N) is a normalisation for M if, and only if, for all $u, v \in S^*$, the following conditions hold [63, Lemma 2.4]:

$$\pi(N(u)) = \pi(u) \qquad \text{and} \qquad \pi(u) = \pi(v) \;\Rightarrow\; N(u) = N(v).$$

Moreover, normalisations (S, N) are is bijections with sections $\iota : M \rightarrowtail S^*$ of π satisfying $\|\iota(u)\| \leqslant \|u\|$ through the formulas [63, Proposition 2.6]

$$N(u) = \iota\pi(u), \quad \text{and} \quad \iota(x) = N(u), \text{ where } u \in S^* \text{ is any representative of } x \in M.$$

6.2.2. Normalisations for non-graded monoids. The definition of normalisation implies that, if (S, N) is a normalisation for a monoid M, then M is graded. To extend the notion to non-graded monoids, we say that, if (S, N) is a normalisation, an element e of S is N-*neutral* if

$$N(ue) = N(eu) = N(u)e$$

hold for every u in S^*. If M is a monoid, we say that (S, N) is a normalisation *mod* e for M if e is an N-neutral element of S and

$$M \simeq \langle S \,|\, (N(u) = u)_{u \in S^*},\; e = 1 \rangle^+ .$$

Normalisations mod e are further developed in [63, §2.2], and they are adapted to present filtered monoids. There, it is proved that normalisations mod e for a filtered monoid M correspond to normalisations for the graded monoid $M \times \mathbb{N}$, through the correspondence $\widetilde{N}(u) = N(u)e^{\|u\| - \|N(u)\|}$. So, there is no loss of generality for considering only the graded case in this presentation.

6.2.3. Quadratic normalisations. A normalisation (S, N) is called *quadratic* if the following conditions hold for every $u \in S^*$:

(i) u is N-normal if, and only if, every length-two factor of u is,

(ii) there exist natural numbers n and $i_1, \ldots, i_n \geqslant 1$ such that $N(u) = \overline{N}_{i_1 \cdots i_n}(u)$.

If (S, N) is a quadratic normalisation for a monoid M, then the definition implies that N is idempotent and

$$M \simeq \left\langle S \,\middle|\, (\overline{N}(st) = st)_{s,t \in S} \right\rangle^+ .$$

6.2.4. Example. Assume that S is a totally ordered set. For u in S^*, define $N(u)$ to be the element of S^* obtained by permuting letters in u that is minimal for the lexicographic order. Then (S, N) is a quadratic normalisation for the free commutative monoid over S. Indeed, $u \in S^*$ is N-normal if, and only if, all its length-two factors are of the form st with $s \leqslant t$, so **(i)** is satisfied. Moreover, **(ii)** holds, since every $u \in S^*$ can be transformed into $N(u)$ by switching adjacent letters that are not in the correct order.

6.2.5. Quadratic normalisations and polygraphs. A 2-polygraph is called *quadratic* if the sources and targets of its 2-cells are of length 2. Quadratic normalisations are related to normalising and confluent presentations of monoids by the following constructions, that are inverses of one another [63, Proposition 3.7]:

(i) If (S, N) is a quadratic normalisation for a monoid M, then $(S \mid X_N)$ is a quadratic, reduced, normalising and confluent presentation of M, where X_N contains a 2-cell

$$st \Rightarrow N(st)$$

for all s and t in S such that st is not N-normal.

(ii) If $(S \mid X)$ is a quadratic, reduced, normalising and confluent presentation of a monoid M, we obtain a quadratic normalisation (S, N_X) for M by putting

$$N_X(u) = \widehat{u}.$$

Note that the 2-polygraph associated to a quadratic normalisation does not always terminate, as shown in §6.5.

6.3. Classes of quadratic normalisations

6.3.1. The left-class and right-class. Let (S, N) be a quadratic normalisation. For n a natural number, we say that (S, N) is of *left-class* n (resp. *right-class* n) if $N(u) = \overline{N}_{12[n]}(u)$ (resp. $N(u) = \overline{N}_{21[n]}(u)$) holds for every u in S^3. For natural numbers m and n, we say that (S, N) is of *class* (m, n) if it is of left-class m and right-class n.

The *minimal left-class of* (S, N) is the smallest natural number n such that (S, N) is of left-class n, if such an n exists, and ∞ otherwise. The *minimal right-class of* (S, N) is defined symmetrically, and the *minimal class of* (S, N) is the pair formed by its minimal left-class and right-class.

6.3.2. Example. Let (S, N) be the lexicographic normalisation of Example 6.2.4, with $|S| \geqslant 2$. For all $r, s, t \in S$, $\overline{N}_{121}(rst)$ and $\overline{N}_{212}(rst)$ are N-normal, so (S, N) is of class $(3, 3)$. Moreover, for $s < t$, we find $\overline{N}_{12}(tts) = tst$ and $\overline{N}_{21}(tss) = sts$, so $(3, 3)$ is the minimal class of (S, N).

6.3.3. Lemma. *Assume that* (S, N) *is a quadratic normalisation.*

(i) *If* $u \in S^*$ *is of length* 3*, then* $N(u) = \overline{N}_{12[n]}(u)$ *implies* $N(u) = \overline{N}_{12[n+1]}(u)$.

(ii) *If* (S, N) *is of left-class* n*, then it is of left-class* m *for every* $m \geqslant n$*, and of right-class* m *for every* $m > n$.

(iii) *The minimal class of* (S, N) *is either of the form* (m, n) *with* $|m - n| \leqslant 1$*, or* (∞, ∞).

Proof. **(i)** Because (S, N) is quadratic, every length-two factor of $N(u)$ is N-normal, so $\overline{N}_1 N(u) = \overline{N}_2 N(u) = N(u)$. So, $N(u) = \overline{N}_{12[n]}(u)$ implies $\overline{N}_{12[n+1]}(u) = \overline{N}_{12[n]}(u)$.

(ii) Assume that (S, N) is of left-class n. Then the previous point implies that $N(u)$ is equal to $\overline{N}_{12[n+1]}(u)$ for every $u \in S^3$, so (S, N) is of left-class $n + 1$ as well and, from there, it is of left-class m for every $m \geqslant n$. For u in S^3, the definition of normalisation and the assumption give $N(u) = \overline{N}_{12[n]}(\overline{N}_2(u)) = \overline{N}_{21[n+1]}(u)$. Hence (S, N) is of right-class $n + 1$ and, from there, of right-class m for every $m > n$.

(iii) If the minimal left-class of (S, N) is a finite number n, then the previous point implies that (S, N) is of right-class $n+1$. Hence the minimal right-class p of (S, N) satisfies $p \leqslant n+1$ and, for symmetric reasons, we have $n \leqslant p+1$. $\square$

6.3.4. Proposition. *Let* (S, N) *be a quadratic normalisation. Then* (S, N) *is*

(i) *of left-class* n *if, and only if,*

$$\overline{N}_{12[n]} = \overline{N}_{12[n+1]} = \overline{N}_{21[n+1]}, \tag{6.1}$$

(ii) *of right-class* n *if, and only if,*

$$\overline{N}_{21[n]} = \overline{N}_{21[n+1]} = \overline{N}_{12[n+1]}, \tag{6.2}$$

(iii) *of class* (n, n) *if, and only if,*

$$\overline{N}_{12[n]} = \overline{N}_{21[n]}. \tag{6.3}$$

Proof. **(i)** If (S, N) is of left-class n, then, for every $u \in S^3$, Lemma 6.3.3 gives

$$N(u) = \overline{N}_{12[n]}(u) = \overline{N}_{12[n+1]}(u) = \overline{N}_{21[n+1]}(u).$$

Conversely, assume (6.1). For $u \in S^3$, if n is odd, the idempotence of $\overline{N}$ and (6.1) imply

$$\overline{N}_1\overline{N}_{12[n]}(u) = \overline{N}_1\overline{N}_1\overline{N}_{12[n-1]}(u) = \overline{N}_1\overline{N}_{12[n-1]}(u) = \overline{N}_{12[n]}(u),$$
$$\overline{N}_2\overline{N}_{12[n]}(u) = \overline{N}_{12[n+1]}(u) = \overline{N}_{12[n]}(u).$$

So $\overline{N}_{12[n]}(u)$ is N-normal for every $u \in S^3$, which means that (S, N) is of left-class n. The proof is symmetric for n even.

(ii) The proof is obtained from the one of **(i)** by symmetry.

(iii) If (S, N) is of class (n, n), then (6.1) and (6.2) hold and imply (6.3). Conversely, assume (6.3) and let $u \in S^3$. Apply (6.3) to u, $\overline{N}_1(u)$ and $\overline{N}_2(u)$, and use the idempotence of $\overline{N}_1$ and $\overline{N}_2$ to obtain

$$\overline{N}_{12[n]}(u) = \overline{N}_{21[n]}(u),$$
$$\overline{N}_{12[n]}(u) = \overline{N}_{12[n]}\overline{N}_1(u) = \overline{N}_{21[n]}\overline{N}_1(u) = \overline{N}_{12[n+1]}(u),$$
$$\overline{N}_{21[n+1]}(u) = \overline{N}_{12[n]}\overline{N}_2(u) = \overline{N}_{21[n]}\overline{N}_2(u) = \overline{N}_{21[n]}(u).$$

So (6.1) and (6.2) are satisfied. $\square$

6.3.5. Higher classes. For $p \geqslant 3$, we say that a quadratic normalisation (S, N) is of *left*-p-*class* n (resp. *right*-p-*class* n) if, for every $u \in S^p$,

$$N(u) = N_{12[n]}^{(p-1)}(u) \qquad \text{(resp. } N(u) = N_{21[n]}^{(p-1)}(u)\text{)}.$$

We say that (S, N) is of p-*class* (m, n) if it is of left-p-class m and right-p-class n. Proposition 6.3.4 has a variant for the p-class, with a similar proof [63, Proposition 3.20].

6.3.6. Example. Consider the lexicographic normalisation (S, N) of Example 6.2.4. We saw in Example 6.3.2 that, for $|S| \geqslant 2$, the minimal (3-)class is $(3, 3)$. For every $p \geqslant 4$ and for every $u \in S^p$, the elements $N_{212}^{[p-1]}(u)$ and $N_{121}^{[p-1]}(u)$ of S^* are well-ordered for the lexicographic order, hence N-normal. Thus (S, N) is of p-class $(3, 3)$.

6.4. Quadratic normalisations of class $(4, 3)$

6.4.1. The domino rule. Assume that S is a set and $\Phi : S^2 \to S^2$ is an idempotent map. For $s, t \in S$, if $\Phi(st) = st$, we draw a small arc such as

$$\begin{tikzcd} \cdot \arrow[r, "s"] & \cdot \arrow[r, "t"] & \cdot \end{tikzcd}$$

and if $s', t' \in S$ satisfy $s't' = \Phi(st)$, we draw a square such as

$$\begin{tikzcd} \cdot \arrow[r, "s'"] \arrow[d, "s"'] & \cdot \arrow[d, "t'"] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$$

We say that Φ *satisfies the domino rule* if, in every situation

$$\begin{tikzcd} \cdot \arrow[r, "s'_1"] \arrow[d, "t_0"'] & \cdot \arrow[r, "s'_2"] \arrow[d, "t_1"] & \cdot \arrow[d, "t_2"] \\ \cdot \arrow[r, "s_1"'] & \cdot \arrow[r, "s_2"'] & \cdot \end{tikzcd} \tag{6.4}$$

the three plain arcs imply the dotted one.

6.4.2. Proposition. *A quadratic normalisation* (S, N) *is of class* $(4, 3)$ *if, and only if,* $\overline{N}$ *satisfies the domino rule.*

Proof. Assume that (S, N) is of right-class 3, and consider the situation (6.4). By definition of the right-class, and because $s_1 s_2$ is N-normal,

$$N(t_0 s_1 s_2) = \overline{N}_{212}(t_0 s_1 s_2) = \overline{N}_{12}(t_0 s_1 s_2) = \overline{N}_2(s'_1 t_1 s_2) = s'_1 s'_2 t_2.$$

So $s'_1 s'_2 t_2$ is N-normal, which implies that $s'_1 s'_2$ is N-normal since (S, N) is quadratic.

Conversely, assume that $\overline{N}$ satisfies the domino rule. Fix $t_0r_1r_2 \in S^3$, and put

$$s_1s_2 = N(r_1r_2), \qquad s'_1t_1 = N(t_0s_1), \qquad s'_2t_2 = N(t_1s_2).$$

Then s'_2t_2 is N-normal by construction, and $s'_1s'_2$ is N-normal by the domino rule, so $s'_1s'_2t_2$ is N-normal because (S, N) is quadratic. So $N(u) = \overline{N}_{212}(u)$ holds for every $u \in S^3$, and (S, N) is of class $(4, 3)$. □

6.4.3. Notation. We denote by $\equiv$ the congruence on $(\mathbb{N}_+)^*$, the free monoid over nonzero natural numbers, generated by the relations

$$\begin{gathered} nn \equiv n, \\ n(n+1)n(n+1) \equiv (n+1)n(n+1)n \equiv (n+1)n(n+1), \\ mn \equiv nm \quad \text{if } |m-n| \geqslant 2. \end{gathered}$$

Note that $\mathbb{N}_+^*/\equiv$ is a variant with infinitely many generators of the monoid M_n of [130]. Let sh : $\mathbb{N}_+^* \to \mathbb{N}_+^*$ be the morphism of monoids induced by $n \mapsto n+1$. For $n \geqslant 1$, put $\gamma_n = 12\cdots n$ and $\gamma_n^{o} = n\cdots 21$, and then define the element δ_n of $\mathbb{N}_+^*$ by induction on n:

$$\delta_1 = \varepsilon \qquad \text{and} \qquad \delta_{n+1} = \mathrm{sh}(\delta_n)\gamma_n.$$

So, the first values of δ_n are $\delta_2 = 212$, $\delta_3 = 323123$, and so on. The element δ_n satisfies, for every $1 \leqslant k < n$, the relation [63, Lemmas 4.11 and 4.12]

$$\delta_n k \equiv \delta_n \equiv k\delta_n. \tag{6.5}$$

6.4.4. Proposition. *Assume that* (S, N) *is a quadratic normalisation of class* $(4, 3)$. *Then, for every* $s \in S$ *and every* N*-normal* $u \in S^n$, *we have*

$$N(tu) = \overline{N}_{\gamma_n}(tu). \tag{6.6}$$

As a consequence, for every $n \geqslant 1$ *and every* $u \in S^n$,

$$N(u) = \overline{N}_{\delta_n}(u). \tag{6.7}$$

Proof. The following diagram summarises the proof of the first assertion:

$$\begin{tikzcd}
\cdot \arrow[r, "s'_1"] \arrow[d, "t_0"'] & \cdot \arrow[r, "s'_2"] \arrow[d, "t_1"] & \cdot \arrow[r, dotted] \arrow[d, "t_2"] & \cdot \arrow[r, "s'_n"] \arrow[d, "t_{n-1}"] & \cdot \arrow[d, "t_n"] \\
\cdot \arrow[r, "s_1"'] & \cdot \arrow[r, "s_2"'] & \cdot \arrow[r, dotted] & \cdot \arrow[r, "s_n"'] & \cdot
\end{tikzcd}$$

Put $t = t_0$ and $u = s_1\cdots s_n$. For every $1 \leqslant k \leqslant n$, define $\overline{N}(t_{n-1}s_n) = s'_nt_n$. By hypothesis on u and construction, all the plain arcs are valid and $\overline{N}_{\gamma_n}(tu) = s'_1\cdots s'_nt_n$. So, as a consequence of the domino rule, all the dotted arcs are also valid, and (6.6) follows. Then (6.7) is deduced by induction on n. □

Before proving the converse direction of Theorem 6.4.7, we state the following consequence of Proposition 6.4.4:

6.4.5. Corollary. *If* (S, N) *is a quadratic normalisation of class* $(4,3)$*, then* (S, N) *is of* p-*class* $(4,3)$ *for every* $p \geqslant 3$.

Proof. Assume $p \geqslant 3$, and let u belong to S^p. By (6.7) and by definition of δ_p, we have

$$N(u) = \overline{N}_{\delta_p}(u) = \overline{N}_{p-1}\overline{N}_{\gamma_{p-2}}\overline{N}_{\mathrm{sh}(\delta_{p-1})}(u)$$

First, we note that $\overline{N}_{\mathrm{sh}(\delta_{p-1})}(u) = N_2^{(p-1)}(u)$. Now, write $u = sv$ and $N(v) = wt$, with $s, t \in S$, $v \in S^{p-1}$ and $w \in S^{p-2}$. Since wt is N-normal by construction, (6.6) gives

$$N(sw) = \overline{N}_{\gamma_{p-2}}(sw) \qquad \text{and} \qquad N(swt) = \overline{N}_{p-1}(N(sw)t).$$

Since $N(sw)$ is N-normal, we must have $N_1^{(p-1)}(sw) = N(sw)$. Moreover, by definition of (S, N),

$$N_2^{(p-1)}(N(sw)t) = N_2^{(p-1)}\overline{N}_{p-1}(N(sw)t) = N(swt).$$

So, we have

$$N_{212}^{(p-1)}(u) = N_{12}^{(p-1)}(swt) = N_2^{(p-1)}(N(sw)t) = N(swt) = N(sN(v)) = N(u).$$

We conclude that (S, N) is of right-p-class 3, hence of p-class $(4,3)$. $\square$

6.4.6. Proposition. *Let* S *be a set, and* $\Phi : S^2 \to S^2$ *be an idempotent map satisfying*

$$\Phi_{212} = \Phi_{2121} = \Phi_{1212}. \tag{6.8}$$

Then, putting $\Phi^*(s) = s$ *for* $s \in S$ *and* $\Phi^*(u) = \Phi_{\delta_n}(u)$ *for* $u \in S^n$ *defines a quadratic normalisation* (S, Φ^*) *of class* $(4,3)$ *such that* $\Phi = \overline{\Phi^*}$.

Proof. Assume that Φ is idempotent and satisfies (6.8). We first observe that $u \equiv v$ implies $\Phi_u = \Phi_v$ for all $u, v \in \mathbb{N}_+^*$. Indeed, $\Phi_{nn} = \Phi_n\Phi_n = \Phi_n$ because Φ is idempotent. The relations

$$\Phi_{n(n+1)n(n+1)} = \Phi_{(n+1)n(n+1)n} = \Phi_{(n+1)n(n+1)}$$

follow from (6.8). Finally, Φ_m and Φ_n commute if $|m - n| \geqslant 2$ because they act on different factors.

Now, let us prove that (S, Φ^*) is a normalisation. We have $\|\Phi^*(u)\| = \|u\|$, and $\Phi^*(s) = s$ for $s \in S$ by definition of Φ^*. Fix $u \in S^m$, $v \in S^n$ and $w \in S^p$. By definition of Φ^*,

$$\Phi^*(u\Phi^*(v)w) = \Phi_{\delta_{m+n+p}}(u\Phi_{\delta_n}(v)w) = \Phi_{\mathrm{sh}^p(\delta_n)\delta_{m+n+p}}(uvw).$$

Since $\mathrm{sh}^p(\delta_n)$ is a product of elements of $\{p+1, \cdots, p+n\}$, we deduce from (6.5) that $\mathrm{sh}^p(\delta_n)\delta_{m+n+p} = \delta_{m+n+p}$. As a consequence,

$$\Phi^*(u\Phi^*(v)w) = \Phi_{\delta_{m+n+p}}(uvw) = \Phi^*(uvw).$$

Let us check that (S, Φ^*) is quadratic. The second condition follows from the definition of Φ^* as $\Phi^*(u) = \Phi_{\delta_n}(u)$ for $u \in S^n$. Fix $u \in S^n$. If u is Φ^*-normal, then, using (6.5), we obtain, for every $1 \leqslant k < n$,

$$\Phi_k(u) = \Phi_k \Phi_{\delta_n}(u) = \Phi_{\delta_n k}(u) = \Phi_{\delta_n}(u) = u,$$

so every length-two factor of u is N-normal. Conversely, if $\Phi_k(u) = u$ holds for every $1 \leqslant k < n$, then, decomposing Φ_{δ_n} into a composite of maps Φ_k gives $\Phi^*(u) = u$.

Finally, since $\overline{\Phi^*} = \Phi$, the quadratic normalisation (S, Φ^*) satisfies (6.2), so that it is of right-class 3 by Proposition 6.3.4, and thus of class $(4, 3)$. $\square$

Propositions 6.4.4 and 6.4.6 give the two directions of

6.4.7. Theorem. *If (S, N) is a quadratic normalisation of class $(4, 3)$, then the restriction $\overline{N}$ of N to S^2 is idempotent and satisfies $\overline{N}_{212} = \overline{N}_{2121} = \overline{N}_{1212}$. Conversely, if φ is an idempotent map on S^2 that satisfies $\varphi_{212} = \varphi_{2121} = \varphi_{1212}$, there exists a quadratic normalisation (S, N) of class $(4, 3)$ satisfying $\varphi = \overline{N}$.*

6.5. Class and termination

6.5.1. Proposition. *If (S, N) is a quadratic normalisation of class $(3, 3)$ for a monoid M, then the associated 2-polygraph $(S \,|\, X_N)$ is a convergent presentation of M. Moreover, for every $u \in S^* \setminus \{1\}$, every 2-cell a of source u in $(S \,|\, X_N)^*$ satisfies*

$$\|a\| \leqslant \frac{\|u\| \, (\|u\| - 1)}{2}.$$

Proof. By construction, the rewriting steps of $(S \,|\, X_N)$ are the 2-cells

$$u \Rightarrow \overline{N}_k(u)$$

for $u \in S^n$ and $1 \leqslant k < n$ such that $\overline{N}_k(u) \neq u$. By Proposition 6.3.4, (S, N) satisfies (6.3), so the 2-cells of $(S \,|\, X_N)$ are of the form

$$u \Rightarrow \overline{N}_w(u),$$

for w an element of the quotient $\overline{B}_n^+$ of the braid monoid B_n^+ by the relations $kk = k$. Since $\overline{B}_n^+$ is finite, we have that $(S \,|\, X_N)$ terminates. Moreover, the length of every element of $\overline{B}_n^+$ is bounded by $n(n-1)/2$. $\square$

6.5.2. Example (The plactic monoid). If X is a totally ordered finite set, the *plactic monoid over* X is the monoid P_X generated by X and subject to the relations

$$\begin{aligned} xzy &= zxy, & &\text{for } x \leqslant y < z, \\ yxz &= yzx, & &\text{for } x < y \leqslant z. \end{aligned}$$

Define a *column of* P_X as a strictly decreasing product of elements of X. A pair (c, c') of columns of P_X is called *normal* if $\|c\| \geqslant \|c'\|$ holds and, for every $1 \leqslant k \leqslant \|c'\|$, the k^{th} element of c is bounded above by the one of c'. A *tableau of* P_X is a product $c_1 \cdots c_n$ of columns of P_X such that (c_i, c_{i+1}) is normal for every $1 \leqslant i < n$.

We refer to [44] for a recent reference on the following facts. The monoid P_X admits its set S of columns as a generating set, and every fibre of $S^* \twoheadrightarrow P_X$ contains a unique tableau of minimal length. More precisely, P_X is isomorphic to its set of tableaux, equipped with a product given by Schensted's insertion algorithm.

We consider the normalisation (S, N) where N maps a product of columns to the corresponding tableau. Since N can be computed locally by the insertion algorithm, progressively replacing each product of two columns cc' by the corresponding tableau $N(cc')$, and by definition of tableaux, (S, N) is quadratic. From [29, §§4.2–4.4], we deduce that, if $|X| \geqslant 2$, then (S, N) is of class $(3, 3)$. By Proposition 6.5.1, we recover [44, Theorem 3.4] that states that the 2-polygraph associated to (S, N) is a convergent presentation of P_X.

A similar argument leads to an infinite convergent quadratic presentation of P_X in terms of *rows*, which are nondecreasing products of elements of X. The proof that the class is $(3, 3)$ is given in [29, §§3.2–3.4].

We refer to [63, Proposition 5.4] for the technical proof of the following result, that relies on a direct analysis of all the possible transformations induced by a quadratic normalisation of class $(4, 3)$. The proof of Proposition 6.5.1 cannot be adapted directly since Krammer's monoid, which would replace $\overline{B}_n^+$ here, is infinite [130].

6.5.3. Theorem. *If* (S, N) *is a quadratic normalisation of class* $(4, 3)$*, then the associated 2-polygraph* $(S \,|\, X_N)$ *is convergent. More precisely, every rewriting sequence from an element of* S^p *has length at most* $2^p - p - 1$.

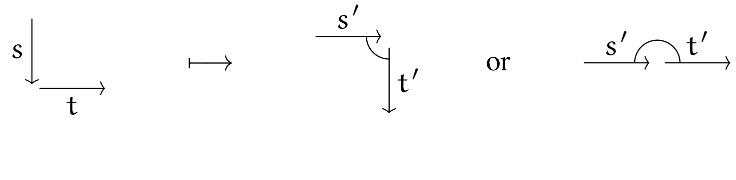
Proof. Let $F(p) \in \overline{\mathbb{N}}$ denote the maximal length of composable sequences of rewriting steps of $(S \,|\, X_N)$ with source in S^p. We prove the inequality $F(p) \leqslant 2^p - p - 1$ by induction on $p \geqslant 2$. For $p = 2$, we have $F(p) \leqslant 1$ by idempotence of N. Assume $p \geqslant 3$ and consider a sequence $(u_k)_{0 \leqslant k \leqslant n}$ of elements of S^p such that there exists a rewriting step from u_k to u_{k+1}. We define $\widehat{S} = S \sqcup \overline{S}$, where $\overline{S}$ contains one element $\bar{s}$ for each $s \in S$. In diagrams, we draw $s \in S$ horizontally, and $\bar{s}$ vertically. Let $\pi : \widehat{S}^* \twoheadrightarrow S^*$ be the projection that identifies $\bar{s}$ and s.

From $(u_k)_k$, we construct, by induction on k, a sequence $(\widehat{u}_k)_k$ of elements of $\widehat{S}^p$ such that

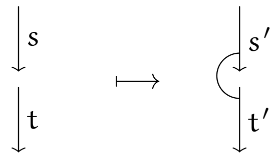
(i) $\pi(\widehat{u}_k) = u_k$,

(ii) if $\bar{s}t$ or $\bar{s}\bar{t}$ is a length-two factor of $\widehat{u}_k$, then st is N-normal.

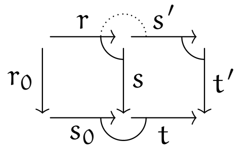
We start by $\widehat{u}_0 = s_1 \cdots s_{p-1}\bar{s}_p$ if $u_0 = s_1 \cdots s_p$, which satisfies **(i)** and **(ii)**. Now, assume that $\widehat{u}_{k-1}$ is defined, for $k \geqslant 1$, and satisfies **(i)** and **(ii)**. By hypothesis, there exists an $i \in \{1, \ldots, p-1\}$ such that $u_k = \overline{N}_i(u_{k-1})$. Let s and t be the letters of u_{k-1} in positions i and $i+1$, and put $s't' = N(st)$. The length-two factor of $\widehat{u}_{k-1}$ at position i is either st, $\bar{s}t$, $s\bar{t}$ or $\bar{s}\bar{t}$. Since st is not N-normal by assumption, **(ii)** prevents $\bar{s}t$ and $\bar{s}\bar{t}$. In the case $s\bar{t}$, we define $\widehat{u}_k$ from $\widehat{u}_{k-1}$ by replacing, at position i, the factor $s\bar{t}$ by $\bar{s}'t'$ if $i < p-1$, or by $\bar{s}'\bar{t}'$ if $i = p-1$. Diagrammatically, this means the following replacement in $\widehat{u}_{k-1}$:



In the case st, we define $\widehat{u}_k$ from $\widehat{u}_{k-1}$ by replacing, at position i, the factor st with $s't'$:



In both cases, by construction, $\widehat{u}_k$ satisfies **(i)**, and **(ii)** is satisfied at every position, except possibly at position $i-1$, if the corresponding letter is some $\overline{r} \in \overline{S}$, i.e. the length-two factor of $\widehat{u}_{k-1}$ starting at position $i-1$ is $\overline{r}s$. By construction of the sequence $\widehat{u}_0, \ldots, \widehat{u}_{k-1}$, this factor $\overline{r}s$ must come from an earlier replacement of some $r_0\overline{s}_0$, i.e. $rs = N(r_0s_0)$. So, because (S, N) is of class $(4, 3)$, the domino rules applies, concluding that rs' is N-normal, so that **(ii)** is also satisfied at position $i-1$:



To conclude the proof, we count the number of transformations $s\overline{t} \mapsto \overline{s}'t'$ (or $s\overline{t} \mapsto \overline{s}'\overline{t'}$) and $st \mapsto s't'$ in the sequence $(\widehat{u}_k)_k$: there are at most $p(p-1)/2$ transformations of the first type, and $F(2) + \cdots + F(p-1)$ transformations of the second type, see [63, Proposition 5.4] for the complete classification. Then, using the induction hypothesis $F(q) \leqslant 2^q - q - 1$ for every $2 \leqslant q < p$, we obtain the required bound. $\square$

6.5.4. Corollary. *Let $(S\,|\,X)$ be a reduced quadratic 2-polygraph. Define $\Phi : S^2 \to S^2$ by $\Phi(u) = v$ if X contains a 2-cell $u \Rightarrow v$, and $\Phi(u) = u$ otherwise. Suppose that, for all $r, s, t \in S$, with rs not reduced, we have*

(i) *if st is reduced, then $\Phi_{12}(rst)$ is reduced,*
(ii) *if st is not reduced, then $\Phi_{1212}(rst) = \Phi_{212}(rst)$.*

Then $(S\,|\,X)$ is convergent.

6.5.5. Proposition. *There exists a quadratic normalisation of class $(4, 4)$ whose associated 2-polygraph does not terminate.*

Proof. The 2-polygraph

$$X = \big(x, y, y', y'', z, z', z'', t \,\big|\, xy \Rightarrow xy',\ yz' \Rightarrow y''z'',\ y'z \Rightarrow y''z'',\ y'z' \Rightarrow yz,\ zt \Rightarrow z't\big)$$

is quadratic and reduced, but does not terminate, because of the following non-trivial closed 2-cell of X^*:

$$\underline{xy}zt \Rightarrow xy'\underline{zt} \Rightarrow x\underline{y'z'}t \Rightarrow xyzt.$$

We refer to [63, Proposition 5.7] for a proof that X is normalising and confluent, and that its associated quadratic normalisation is of minimal class $(4, 4)$. $\square$

6.5.6. Example (The Chinese monoid). For a totally ordered finite set X, the *Chinese monoid over* X is the monoid C_X generated by X and submitted to the relations

$$zyx = zxy = yzx$$

for all $x \leqslant y \leqslant z$ in X [46].

Assume that X has three elements and denote by S the set obtained from X by adjoining the three 1-cells yx for $x < y$, and yy if y is the middle element of X (neither the minimal one nor the maximal one). We denote by $|$ the product in S^* to distinguish it from the product of C_X. The following twelve 2-cells are derivable from the defining relations of C_X:

(i) the nine 2-cells $y|x \Rightarrow yx$, $y|yx \Rightarrow yx|y$, and $yx|x \Rightarrow x|yx$, for $x < y$ in X,

(ii) the two 2-cells $y|zx \Rightarrow zx|y$ and $z|yx \Rightarrow zx|y$, for $x < y < z$ in X,

(iii) the 2-cell $y|y \Rightarrow yy$, if y is the middle element of X.

The 2-polygraph so obtained terminates (using the weighted right-lexicographic order generated by $x < yx$ for $x \leqslant y$ and $zx < y$ for $x < y$) and, after application of Knuth-Bendix completion, it yields a reduced convergent quadratic presentation of C_X. The corresponding normalisation is of minimal class $(4, 4)$, the worst case being reached on $z|yy|y$ if y is the middle element and $z > y$ holds.

Similar convergent quadratic presentations also exist when X has four or five elements, to be compared with the nonquadratic ones of [45, Theorem 3.3]. The class is $(5, 4)$ in both cases.

6.6. Garside normalisations

6.6.1. Garside families. Assume that M is a monoid. We denote by $\preccurlyeq$ the *left-divisibility relation of* M, defined for all $x, y \in M$ by $x \preccurlyeq y$ if $xx' = y$ for some $x' \in M$. Fix $S \subseteq M$. A pair (s, t) of elements of S is called *normal* if it satisfies, for all $r \in S$ and $x \in M$,

$$r \preccurlyeq xst \;\Rightarrow\; r \preccurlyeq xs.$$

An element u of S^* is called *normal* if, writing $u = s_1 \cdots s_n$ with each s_k in S, every pair (s_k, s_{k+1}) is normal.

Assume that M has no nontrivial invertible element, and S contains 1. We say that S is a *Garside family* if every element x of M has a normal representative in S^*. This definition is a restricted case of the general definition of a Garside family developed in [60]. If M is also left-cancellative, then Propositions III.1.25 and III.1.30 of [60] imply that, if $S \subseteq M$ is a Garside family, then every element of M admits a unique normal representative of minimal length in S^*. We denote by $N^S : S^* \to S^*$ the corresponding map.

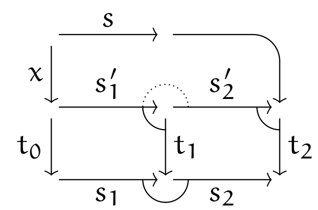
6.6.2. Left-weighted normalisations. Assume that (S, N) is a (quadratic) normalisation for a monoid M. We say that (S, N) is *left-weighted* if, for all $s, t, s', t' \in S$, the equality $s't' = N(st)$ implies $s \preccurlyeq s'$ in M.

6.6.3. Theorem. *Let* M *be a left-cancellative monoid, with no nontrivial invertible element, and* (S, N) *be a quadratic normalisation for* M. *The following assertions are equivalent:*

(i) S *is a Garside family, and* $N = N^S$.

(ii) (S, N) *is of class* $(4, 3)$ *and left-weighted.*

Proof. Assume that S is a Garside family and $N = N^S$. The pair (S, N^S) is a normalisation by definition of N^S. Moreover, $u \in S^*$ is N^S-normal if, and only if, every length-two subfactor of u is N^S-normal. We prove the second axiom of a quadratic normalisation together with the fact that (S, N^S) is of class $(4, 3)$ by checking that $\overline{N^S}$ satisfies the domino rule. Consider a situation as in (6.4), and fix $s \in S$ and $x \in X$ such that $s \preccurlyeq x s'_1 s'_2$:



We have $s \preccurlyeq x s'_1 s'_2 t_2 = x t_0 s_1 s_2$. Because $s_1 s_2$ is normal, we have $s \preccurlyeq x t_0 s_1 = x s'_1 t_1$. The latter being also normal, we deduce $s \preccurlyeq x s'_1$, so $s'_1 s'_2$ is normal. Finally, fix $s, t, s', t' \in S$ such that $s't' = N(st)$. By hypothesis, the relation $st = s't'$ holds in M, so $s \preccurlyeq s't'$. Since $s't'$ is normal, we deduce $s \preccurlyeq s'$, so (S, N) is left-weighted.

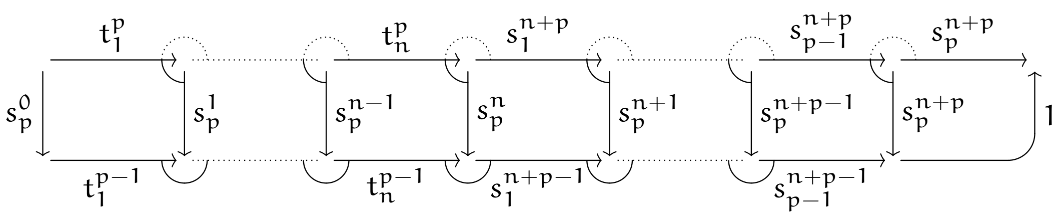
Conversely, assume that (S, N) is of class $(4, 3)$ and left-weighted. To prove that S is a Garside family, we use Proposition IV.2.7 of [60]: if S contains 1 and is closed under right-divisor, then S is a Garside family if, and only if, every $x \in M \setminus \{1\}$ admits a maximal left-divisor in S. We have $1 \in S$ by hypothesis, so let us check that S is closed under right-divisor. Fix $s \in S$ and x, y in M such that $s = xy$ holds in M, and prove that $y \in S$. If $y = 1$, or if $x = 1$ and thus $y = s$, then $y \in S$. Assume $x, y \neq 1$, choose representatives u and v of x and y in S^*, and put

$$N(u) = s^0_m \cdots s^0_1 \qquad \text{and} \qquad N(v) = t^0_1 \cdots t^0_n.$$

We define the elements s^i_p, for $1 \leqslant p \leqslant m$ and $1 \leqslant i < m + n - 1$, and t^j_q, for $1 \leqslant q \leqslant n$ and $1 \leqslant j \leqslant m$, by induction thanks to the formulas

$$\begin{aligned} t^p_q s^q_p &= N(s^{q-1}_p t^{p-1}_q) && \text{for } 1 \leqslant p \leqslant m \text{ and } 1 \leqslant q \leqslant n, \\ s^{n+q-1}_p s^{n+p}_q &= N(s^{n+p-1}_q s^{n+q-2}_p) && \text{for } 1 \leqslant p < q \leqslant m. \end{aligned}$$

For $p \in \{1, \ldots, m\}$ the following figure illustrates a part of these definitions:



Since (S, N) is of class $(4, 3)$, it satisfies the domino rule, and so the upper row of this figure is N-normal, provided the bottom row is. We conclude that

$$N(uv) = t^m_1 \cdots t^m_n s^{m+n-1}_1 \cdots s^{m+n-1}_m.$$

But, by hypothesis, we have $N(uv) = N(s) = s$, so

$$t_1^m = s \qquad \text{and} \qquad t_2^m = \cdots = t_n^m = s_1^{m+n-1} = \cdots = s_m^{m+n-1} = 1.$$

We deduce that, for all $1 \leqslant p < q \leqslant m$, $N(s_q^{n+p-1} s_p^{n+q-2}) = 1$. But M has no nontrivial invertible element, so $s_p^q = 1$ for all $1 \leqslant p \leqslant m$ and $n \leqslant q < m+n$. Similarly, we have $N(s_p^{q-1} t_q^{p-1}) = 1$ for all $1 \leqslant p \leqslant m$ and $1 < q \leqslant n$, from which we obtain, in particular, $t_2^0 = \cdots = t_n^0 = 1$. We conclude $y = t_1$, so that $y \in S$.

Now, fix $x \in M \setminus \{1\}$, choose a representative u in S^* and put $N(u) = s_1 \cdots s_n$ with each s_i in S. By construction, s_1 is a left-divisor of x in M. Assume that $t \in S$ is a left-divisor of M, writing $x = ty$. If $y = 1$, then $x = t$ so $s_1 = t$. Otherwise, choose a representative v of y and put $N(v) = t_1 \cdots t_p$. Write $t_1' t' = N(tt_1)$. Because (S, N) is of class $(4, 3)$, and $N(u)$ and $N(v)$ are N-normal, then t_1' is the first letter of both $N(u)$ and $N(v)$. So, we deduce, on the one hand, that $t_1' = s_1$, and, on the other hand, that t left-divides t_1' since (S, N) is left-weighted, which concludes the proof that S is a Garside family.

There remains to prove $N^S = N$. Since the two normalisations (S, N) and (S, N^S) are quadratic of class $(4, 3)$, it is sufficient to prove that $N(st) = N^S(st)$ holds for all $s, t \in S$. Fix $s, t \in S$ and put $s't' = N(st)$, so that s' is the maximal left-divisor of st in M, and $s't' = st$ holds in M. Hence, if $r \in S$ is a left-divisor of $s't'$ in M, then r is a left-divisor of s'. By Corollary IV.1.31 of [60], this implies that $N(st)$ is normal, i.e. $N(st) = N^S(st)$. □

We end the chapter with consequences of the results given so far, giving a general setting for monoids to admit presentations such as those considered in §5.5 for Artin monoids and Garside monoids. We begin with the fact that a Garside family always induce a convergent presentation such as the one in the proof of Proposition 5.5.3:

6.6.4. Corollary. *Let M be a left-cancellative monoid with no nontrivial invertible elements, and $S \subseteq M$ be a Garside family. Then M admits, as a convergent presentation, the 2-polygraph with one 1-cell for each nontrivial element of S, and one 2-cell*

$$st \Rightarrow N^S(st)$$

for all nontrivial elements s and t of S such that st is not N^S-normal. In particular, every Artin monoid admits a finite convergent presentation.

Proof. The first part of the result is a direct consequence of Theorems 6.5.3 and 6.6.3. The second part comes from Theorem 1.1 of [61]. □

We refer to [63, Proposition 6.15] for the following generalisation that every Artin monoid and every Garside monoid admits a Garside-like presentation:

6.6.5. Proposition. *Let M be a left-cancellative monoid, and (S, N) is a left-weighted quadratic normalisation of class $(4, 3)$ for M. Then M admits, as a presentation, the 2-polygraph* $\mathrm{Gar}_2(S, N)$ *with one 1-cell for each nontrivial element of S, and one 2-cell*

$$s|t \Rightarrow st,$$

for all nontrivial elements s and t in S whose product st in M lies in S, where $s|t$ denotes the product in S^*.

6.6.6. Example. If W is a Coxeter group, then the associated Artin monoid $\mathrm{B}^+(W)$ has no nontrivial invertible element, and W injects in $\mathrm{B}^+(W)$. The image of W in $\mathrm{B}^+(W)$ is a Garside family, so Corollary 6.6.4 generalises the fact that the Garside generators of $\mathrm{B}^+(W)$ induce a convergent presentation, as proved in Proposition 5.5.3. Moreover, if W is finite, then this Garside family is minimal. So, if $W = \mathrm{S}_3$, the Artin monoid $\mathrm{B}^+(\mathrm{S}_3) = \mathrm{B}_3^+$ admits S_3 as a finite Garside family, and the corresponding quadratic normalisation is of minimal class $(3,3)$.

When W is infinite, there exists a finite subset of W that is a Garside family [61]. Consider, for example, the Artin monoid associated to the infinite Coxeter group $W = \widetilde{\mathrm{A}}_2$:

$$\mathrm{B}^+(\widetilde{\mathrm{A}}_2) \simeq \langle a, b, c \mid aba = bab,\ bcb = cbc,\ cac = aca \rangle^+ .$$

This Artin monoid admits a minimal 16-element Garside family S, which are the vertices of the following diagram where arrows stand for right-divisibility:

$$\begin{tikzcd}
caba & & ba \arrow[dl, bend right] & & ca \arrow[dr, bend left] & & baca \\
& aba \arrow[ul] & & a \arrow[ul, bend right] \arrow[ur, bend left] & & aca \arrow[ur] & \\
& ab \arrow[u] & & 1 \arrow[u] \arrow[dl] \arrow[dr] & & ac \arrow[u] & \\
& & b \arrow[ul, bend left] \arrow[d] & & c \arrow[ur, bend right] \arrow[d] & & \\
& & cb \arrow[dr, bend right] & & bc \arrow[dl, bend left] & & \\
& & & bcb \arrow[d] & & & \\
& & & abcb & & &
\end{tikzcd}$$

The normalisation corresponding to this Garside family is of minimal class $(4,3)$. Indeed, it is not of left-class 3 because $\overline{\mathrm{N}}^{\mathrm{S}}_{121}(a|ab|ac) = aba|b|c$ but $b|c$ is not normal. The convergent presentation of $\mathrm{B}^+(\widetilde{\mathrm{A}}_2)$ of Corollary 6.6.4 has 15 generators and 87 relations.

Chapter 7
Squier theory for associative algebras

We fix a field $\mathbb{K}$ for the whole chapter. We denote by $\mathcal{V}ect$ the category of vector spaces over $\mathbb{K}$, and by $\mathcal{A}lg$ the one of unital and associative algebras over $\mathbb{K}$. When no confusion may arise, we just say associative algebras, or simply algebras, for the objects of $\mathcal{A}lg$.

7.1. Introduction

7.1.1. Context. During the twentieth century, rewriting-like methods have been developed in linear algebra, although with a different vocabulary, and in a somewhat restricted setting. The corresponding concepts have been introduced in particular to compute normal forms for different types of algebras presented by generators and relations, with applications to the decision of the ideal membership problem, and to the construction of bases, such as Poincaré-Birkhoff-Witt bases. For example, Shirshov introduced in [190] an algorithm to compute a linear basis of a Lie algebra presented through generators and relations. He used the notion of composition of elements in a free Lie algebra to describe the critical branchings. He gave an algorithm to compute bases in free algebras having the confluence property, and he proved the composition lemma, which is the analogue of Newman's lemma for Lie algebras. As an application, he deduced a constructive proof of the Poincaré-Birkhoff-Witt theorem.

Rewriting methods to compute with ideals of commutative polynomial rings were also introduced by Buchberger with Gröbner basis theory [42, 41]. Gröbner bases are sets of relations that satisfy the confluence property, plus a constrained form of termination, corresponding to a compatibility with a *monomial order*, i.e. a wellfounded total order on the monomials. Buchberger described critical branchings with the notion of S-polynomial, and gave an algorithm for the computation of Gröbner bases, using a linear counterpart to Newman's lemma, and which is the analogue of Knuth-Bendix's completion procedure for commutative algebras. In the same period, ideas in the spirit of Gröbner bases appear in several other works: by Hironaka and Grauert with standard bases for power series rings [110, 84], or for applications of Newman's lemma for universal algebras by Cohn [56]. The domain took foundation in several works on algorithmic methods in elimination theory by Macaulay with H-bases [155], by Janet with involutive bases [114], or Gunther with notions similar to Gröbner bases [102]. Bokut and Bergman have independently extended Gröbner bases to associative algebras, obtaining the analogue of Newman's lemma for free associative algebras, called respectively the composition lemma and the diamond lemma [28, 27]. Buchberger's algorithm has then been extended to this setting [167, 206]. Subsequently, rewriting methods were developed for a wide range of algebraic structures, such as Weyl algebras [188] or operads [68]. See [167, 205, 35] for a comprehensive treatment on noncommutative Gröbner bases.

At the end of the eighties, through Anick's and Green's works, noncommutative Gröbner bases have found new applications, in the shape of constructive methods to compute free resolutions of associative algebras [5, 6, 7, 85]. Their constructions provide small explicit resolutions to compute homological invariants of algebras presented by generators and a Gröbner basis: homology groups, Hilbert series, and Poincaré series. Anick's resolution consists in a complex generated by Anick's chains, i.e. certain iterated overlaps of the leading terms of the relations as in the later Kobayahi's resolution for monoids, and its differential is obtained by deforming the differential of a complex for an associated monomial algebra. This construction has many applications, such as an algorithm for the computation of Hilbert series [205]. The chains and the differential of the resolution are constructed recursively, making its implementation possible [12], but the differential is complicated to make explicit in general. Sköldberg introduced in [191] a homotopical method based on discrete Morse theory to derive Anick's resolution from the bar resolution.

Anick's resolution found applications in the study of Koszul algebras: a connected graded algebra A is *Koszul* if the Tor groups $\mathrm{Tor}^{A}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq k$, where k is the homological degree, and i comes from the grading of the algebra. For intuition, in a quadratic algebra A, the groups $\mathrm{Tor}^{A}_{k,(i)}(\mathbb{K},\mathbb{K})$ always vanish for $i < k$: the Koszul property corresponds to the case where the Tor is concentrated on the diagonal [26]. Koszul algebras were introduced by Priddy, and he proved that quadratic algebras having a Poincaré-Birkhoff-Witt basis are Koszul [180]. Thus, if A admits a quadratic Gröbner basis, then it is Koszul: indeed, the hypothesis implies the existence of a Poincaré-Birkhoff-Witt basis [85], another proof coming from the fact that Anick's resolution is concentrated in the diagonal [6, 86]. Backelin gave a characterisation of the Koszul property for quadratic algebras in terms of lattices [11, 13], and this condition was later interpreted by Berger's X-confluence [24]. The quadratic Gröbner basis method to prove Koszulness has been extended to the case of operads by Dotsenko and Khoroshkin in [68], and Koszulness was generalised by Berger in [25] to the case of N-homogeneous algebras.

For his results, Berger has introduced a notion of reduction operators, which is an alternative point of view on rewriting, based on linear operators mapping basis elements to their normal form, with similarities to the quadratic normalisations for monoids of Chapter 6. Building on Berger's work, Chenavier has recently developed an original theory of rewriting in associative algebras, alternative to Gröbner bases and to the one presented here, with applications in homological algebra [49]. In particular, he gave an algebraic account of completion in terms of the lattice structure of reduction operators [51] and an interpretation of Faugère's F4 completion algorithm [72] in this setting [50]; he refined Berger's results on the Koszul complex of N-homogeneous algebras [48]; and he obtained a Squier-like result to construct homological syzygies of associative algebras presented by reduction operators [52].

7.1.2. Summary. To adapt the theory and methods developed so far to associative algebras, §7.2 explains how to introduce an analogue of higher categories that provide a homotopical setting for polygraphic resolutions of associative algebras. The structure of ∞-categories works for monoids because the latter can be seen as categories with one 0-cell. However, ∞-categories do not really fit for other algebraic structures, that can have several associative products (e.g.

boolean algebras), or operations that satisfy other compatibility laws than associativity (e.g. Lie algebras). To solve this, observe that ∞-categories with one 0-cell are the same as ∞-categories internal to the category $\mathcal{Mon}$ of monoids, shifting all cells by one dimension and replacing the 0-composition by the monoidal product. Now, the theoretical setting adapts to any other algebraic structure, by replacing $\mathcal{Mon}$ with the adequate category: here, we consider ∞-categories internal to $\mathcal{Alg}$, called ∞-*algebras*, and the underlying structure of ∞-vector space, which is an ∞-category internal to $\mathcal{Vect}$; note that our 1-vector spaces are the 2-vector spaces of [15]. The structures of ∞-vector space and ∞-algebras are rather rich, and in particular contain much redundancy. Indeed, an ∞-vector space is the same as an ∞-groupoid internal to $\mathcal{Vect}$, and as an ∞-globular vector space. This generalises Proposition 2.5 of [124], and is intuitively similar to Bourn's equivalence between chain complexes in an abelian category $\mathcal{A}$ and internal ∞-categories in $\mathcal{A}$ [34, Theorem 3.2]. Here, the main arguments are that the compositions of an ∞-vector space necessarily satisfy the relation $a \star_i b = a - t_i(a) + b$ whenever a and b are i-composable, and every cell a of strictly positive dimension has an inverse, given by $a^- = s(a) - a + t(a)$. For ∞-algebras, the situation is summarised by

Theorem 7.2.4. *An ∞-algebra is the same as an ∞-groupoid internal to $\mathcal{Alg}$, and as a pair (A, M) formed by an associative algebra A and a globular A-bimodule M such that $M_0 = A$ and that satisfies the extra condition $as_0(b)+t_0(a)b-t_0(a)s_0(b) = s_0(a)b+at_0(b)-s_0(a)t_0(b)$.*

For intuition, the extra condition corresponds to the compatibility of the algebra product with the 0-composition, the latter being rewritten using $a \star_0 b = a - t_0(a) + b$. The second part of Theorem 7.2.4 says that the structure of ∞-algebra boils down to a simpler one, useful to describe free ∞-algebras and, as a consequence, the corresponding notion of polygraph, following the same pattern as for ∞-categories. Polygraphs for associative algebras are a special case of the general construction of polygraphs, or computads, for finitary monads on globular sets (here, the monad of ∞-algebras) given in [17, Definition 2.1]. We now say n-polygraph for this notion of polygraph for associative algebras, and set-theoretic n-polygraph for the one considered so far.

Then, §7.3 develops a rewriting theory for associative algebras, the main difficulty coming from the first part of Theorem 7.2.4: in contrast with the case of monoids, there is no difference between a 1-category internal to $\mathcal{Alg}$ and a 1-groupoid internal to $\mathcal{Alg}$, making impossible to consider the 1-cells of the former as rewriting sequences among the 1-cells of the latter, seen as equivalences. To circumvent this obstruction, we isolate, among the 1-cells, a set of positive ones that will play the same role as a free 1-category with respect to the corresponding free 1-groupoid: first, the set of positive 1-cells is big enough for every 1-cell to factor into a composite of positive 1-cells and their opposites; second, the set of positive 1-cells is small enough for preventing a nontrivial 1-cell and its inverse to be positive at the same time, so that termination is possible. From this notion of positive 1-cells, we define branchings and confluence as in the set-theoretic case, but termination is handled differently. For a set-theoretic 2-polygraph X, termination is defined as any one of the equivalent properties: there exist no infinite sequence of composable rewriting steps, or there exists a wellfounded order relation on X_1^* that is compatible with the product and such that $s(\alpha) > t(\alpha)$ holds for each 2-cell α of X. For a 1-polygraph X, the second property implies the first one, but we conjecture that the converse implication does not hold, and we define termination as the second, stronger property. We obtain

Theorem 7.3.6. *Let* A *be an algebra and* X *be a convergent presentation of* A*. Then the set* $\mathrm{Red}_m(X)$ *of reduced monomials of* $\mathbb{K}\langle X\rangle$ *is a linear basis of* A*. As a consequence, the vector space* $\mathrm{Red}(X)$ *of reduced* 0*-cells of* $\mathbb{K}\langle X\rangle$*, equipped with the product defined by* $a \cdot b = \widehat{ab}$*, is an algebra that is isomorphic to* A*.*

The convergent presentations so-defined are strict generalisations of noncommutative Gröbner bases, and, in the case of homogeneous algebras, of Poincaré-Birkhoff-Witt bases, as explained in §7.4. Indeed, noncommutative Gröbner bases correspond to a subclass of convergent presentations of associative algebras: those whose 1-cells are well-oriented with respect to a monomial order. More precisely, from a Gröbner basis $\mathcal{G}$ for an ideal I of a free algebra $\mathbb{K}\langle X\rangle$, with respect to a monomial order $\preccurlyeq$, one can derive a convergent presentation of the quotient algebra $\mathbb{K}\langle X\rangle/I$, whose 1-cells correspond to the replacement of the leading monomials of the elements of $\mathcal{G}$ by their remainders. Moreover, the inclusion of Gröbner bases into convergent presentations is strict, as testified by the 1-polygraph $\big(x, y, z \,\big|\, xyz \to x^3 + y^3 + z^3\big)$ studied in Example 7.4.7. This 1-polygraph is convergent, but the orientation of its only 1-cell is not compatible with any possible monomial order, for which one of the three monomials x^3, y^3 or z^3 is necessarily bigger than xyz. The same strict inclusion holds for Poincaré-Birkhoff-Witt bases in the case of homogeneous N-algebras (generalising Priddy's PBW bases for quadratic algebras with the deglex order [180, Section 5.1]): they correspond to the convergent N-homogeneous presentations of associative algebras whose 1-cells terminate with respect to a monomial order.

In §7.5, we adapt Squier's theorem to associative algebras, with a similar proof strategy as in the set-theoretic case. However, there is a difference in the equivalence between local Y-confluence and critical Y-confluence: indeed, here, this equivalence requires the termination hypothesis, and a more complicated proof. This comes from the fact that, in general, the trivial branchings are not $\emptyset$-confluent without termination, as shown by the two counterexamples of Remark 4.2.4 in [95]. Nevertheless, we still obtain Squier's theorem for algebras:

Theorem 7.5.3. *Let* X *be a convergent presentation of an associative algebra, and* Y *be an extension of* $\mathbb{K}\langle X\rangle$*. If* X *is* Y*-convergent, then* Y *is acyclic. In particular, if* Y *contains a* 2*-cell*

$$\begin{tikzcd} & b \arrow[dr, "h"] & \\ a \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow F"] & & d \\ & c \arrow[ur, "k"'] & \end{tikzcd}$$

for every critical branching (f, g) *of* X*, with* h *and* k *positive* 1*-cells of* $\mathbb{K}\langle X\rangle$*, then* Y *is acyclic.*

Finally, §7.6 summarises the extension of Squier theorem to construct polygraphic resolutions of associative algebras from convergent presentations, and applications in homological algebra. We start by a study of the standard polygraphic resolution $\mathrm{Std}(\mathcal{B})$ of an augmented associative algebra A with a fixed linear basis $\mathcal{B}$. The explicit definition of this polygraph is still unknown in the set-theoretic case, but is possible here thanks to the linear structure of ∞-algebras. The complete proof that $\mathrm{Std}(\mathcal{B})$ is a polygraphic resolution of A can be found in §6.1 of [95], and relies on the construction of a contraction, a notion developed in §5.2 of the same article in analogy with the set-theoretic case. Next, to compute Squier's polygraphic resolution of an augmented algebra, we apply, on the standard polygraphic resolution, a collapsing mechanism that is similar to the one described by Brown in [37], and which is a polygraphic version of algebraic Morse theory for chain complexes [191]. This method is based on the notion of collapsing scheme of a polygraph X, which is a graded partial map $\varphi : X \to X$ that creates a correspondence between n-cells and $(n+1)$-cells of X that can be collapsed together without changing the homotopical properties of X. If A is an augmented algebra with a convergent presentation X, we collapse the standard polygraphic resolution of A, equipped with the linear basis $\mathrm{Red}_m(X)$ of reduced monomials. For that, we identity a notion of critical cell in $\mathrm{Std}(\mathrm{Red}_m(X))$, that, similarly to Anick's chains, correspond to some critical n-branchings, forming a graded subset $\mathrm{Sq}(X)$ of $\mathrm{Std}(\mathrm{Red}_m(X))$. The remaining cells are called subcritical or supercritical, depending on whether an induced n-branching is trivial or not, and we establish a bijective correspondence φ between the two families. The result is

Theorem 7.6.4. *Let A be an augmented algebra and X be a convergent presentation of A. Then φ is a collapsing scheme of $\mathrm{Std}(\mathrm{Red}_m(X))$ onto $\mathrm{Sq}(X)$, and, as a consequence, $\mathrm{Sq}(X)$ is a polygraphic resolution of A.*

As in the set-theoretic case, an abelianisation process turns any polygraphic resolution into a resolution by free bimodules, yielding in particular new sufficient or necessary conditions for homogeneous algebras to be Koszul. Finally, the constructions of this section are applied on two examples, the symmetric and the exterior algebras.

7.2. Higher associative algebras and their polygraphs

7.2.1. Higher vector spaces. Fix $n \in \overline{\mathbb{N}}$. Taking $\mathcal{C} = \mathcal{V}ect$ in §2.2.2 and §2.2.4 gives the categories $n\mathcal{G}lob(\mathcal{V}ect)$ of *n-globular vector spaces* and *globular linear maps*, and $n\mathcal{C}at(\mathcal{V}ect)$, denoted by $n\mathcal{V}ect$, of *n-vector spaces* and *linear n-functors*. If V is an n-globular vector space, then, in the pullback $V \star_k V$, we have $\lambda(a, a') + \mu(b, b') = (\lambda a + \mu b, \lambda a' + \mu b')$. Hence, if V is an n-vector space, the linearity of the k-composition reads, in the same context,

$$(\lambda a + \mu b) \star_k (\lambda a' + \mu b') = \lambda(a \star_k a') + \mu(b \star_k b').$$

7.2.2. Proposition. *Fix $n \in \overline{\mathbb{N}}$. The forgetful functors $n\mathcal{G}pd(\mathcal{V}ect) \to n\mathcal{V}ect \to n\mathcal{G}lob(\mathcal{V}ect)$ are isomorphisms. In particular :*

(i) *An n-globular vector space V can be uniquely extended into an n-vector space, by putting, for all natural numbers $0 \leqslant i < k$ and i-composable pair (a, b) of k-cells of V,*

$$a \star_i b = a - t_i(a) + b. \tag{7.1}$$

(ii) *If V is an n-vector space, then, for every $k \geqslant 1$, every k-cell a of V is invertible, with inverse given by $a^- = s(a) - a + t(a)$.*

Proof. If V is an n-vector space, for (a, b) a k-cell of $V \star_i V$, the linearity of $\star_i$ gives (7.1):

$$\begin{aligned} a \star_i b &= (a - s_i(b) + s_i(b)) \star_i (t_i(a) - t_i(a) + b) \\ &= (a \star_i t_i(a)) - (s_i(b) \star_i t_i(a)) + (s_i(b) \star_i b) = a - t_i(a) + b. \end{aligned}$$

Moreover, if W is an n-globular vector space, then every n-globular linear map $F : V \to W$ is automatically a linear n-functor between the induced n-vector spaces:

$$F(a \star_i b) = F(a - t_i(a) + b) = F(a) - t_i(F(a)) + F(b) = F(a) \star_i F(b).$$

The rest of the proof is a consequence of these facts, see [95, Proposition 1.2.3]. □

7.2.3. Higher associative algebras. Fix $n \in \overline{\mathbb{N}}$. Taking $\mathcal{C} = \mathcal{A}lg$ in §2.2.2 and §2.2.4 gives the categories $n\mathcal{G}lob(\mathcal{A}lg)$ of *n-globular algebras*, and $n\mathcal{C}at(\mathcal{A}lg)$, denoted by $n\mathcal{A}lg$, of *n-algebras*.

For an n-globular algebra A, the product of (a, a') and (b, b') in $A \star_k A$ is given by $(a, a')(b, b') = (ab, a'b')$. This implies that, if A is an n-algebra, the fact that the k-composition is a morphism of algebras is equivalent to

$$ab \star_k a'b' = (a \star_k a')(b \star_k b'). \tag{7.2}$$

If A is an algebra, and $\mathcal{B}imod(A)$ is the category of A-bimodules, an *n-globular A-bimodule* is an object of $n\mathcal{G}lob(\mathcal{B}imod(A))$. The category $n\mathcal{G}lob(\mathcal{B}imod)$ of *n-globular bimodules* is the one whose objects are pairs (A, M) formed by an algebra A and an n-globular A-bimodule M, and whose morphisms from (A, M) to (B, N) are pairs (f, g) made of a morphism $f : A \to B$ of algebras and a morphism $g : M \to N$ of bimodules, i.e. $g(ama') = f(a)g(m)f(a')$ holds for all $a, a' \in A$ and $m \in M$.

7.2.4. Theorem. *Fix $n \in \overline{\mathbb{N}}$. The following categories are isomorphic:*

(i) *The category $n\mathcal{A}lg$ of n-algebras.*

(ii) *The category $n\mathcal{G}pd(\mathcal{A}lg)$ of internal n-groupoids in $\mathcal{A}lg$.*

(iii) *The full subcategory of $n\mathcal{G}lob(\mathcal{A}lg)$ whose objects are the n-globular algebras A that satisfy, for all k-cells a and b of A, with $k \geqslant 1$, the relations*

$$ab = as_0(b) + t_0(a)b - t_0(a)s_0(b) = s_0(a)b + at_0(b) - s_0(a)t_0(b). \tag{7.3}$$

(iv) *The full subcategory of $n\mathcal{G}lob(\mathcal{B}imod)$ whose objects are the pairs (A, M) such that M_0 is equal to A, with its canonical A-bimodule structure, and that satisfy, for all k-cells a and b of M, with $k \geqslant 1$, the relation*

$$as_0(b) + t_0(a)b - t_0(a)s_0(b) = s_0(a)b + at_0(b) - s_0(a)t_0(b). \tag{7.4}$$

Proof. The first observation is that, if A is an n-algebra, a and b are k-cells of A, for $k \geqslant 1$, then (7.2) and (7.1) imply

$$ab = (a \star_0 t_0(a))(s_0(b) \star_0 b) = as_0(b) \star_0 t_0(a)b = as_0(b) - t_0(a)s_0(b) + t_0(a)b,$$

and, symmetrically,

$$ab = (s_0(a) \star_0 a)(b \star_0 t_0(b)) = s_0(a)b \star_0 at_0(b) = s_0(a)b - s_0(a)t_0(b) + at_0(b),$$

so that (7.3) is satisfied. The second observation is that, in an n-globular algebra A that satisfies (7.3), the 0-composition defined by (7.1) automatically satisfies (7.2). Indeed, consider (a, a') and (b, b') of dimension at least 1 in $A \star_k A$, and put $x = t_0(a)$ and $y = t_0(b)$. By (7.1),

$$(a\star_0 a')(b\star_0 b') = (a-x+a')(b-y+b') = ab+ab'+a'b+a'b'-ay-a'y-xb-xb'+xy.$$

Then, using (7.3) on ab' and $a'b$ gives $ab' = ay + xb' - xy$ and $a'b = a'y + xb - xy$, so that

$$(a \star_0 a')(b \star_0 b') = ab - xy + a'b' = ab \star_0 a'b'.$$

We refer to [95, Theorem 1.3.3] for the rest of the proof. □

7.2.5. Extensions of higher algebras. Fix a natural number n, and let A be an n-algebra. An *extension of* A is a set X equipped with maps $s, t : X \to A_n$ such that, for every x in X, the pair $(s(x), t(x))$ is an n-sphere of A. If X is an extension of A, we write $\approx_X$ the congruence relation on the parallel n-cells of A generated by $s(x) \approx_X t(x)$ for every $x \in X$, and we call X *acyclic* if, for every n-sphere (a, b) of A, we have $a \equiv_X b$. Every n-algebra A has two canonical extensions: the empty one, and $n\,\mathrm{Sph}(A)$, containing all the n-spheres of A.

For $n \geqslant 0$, the category $n\mathcal{Alg}^+$ of *extended* n*-algebras* is defined like $n\mathcal{Cat}^+$ in §2.3.3, replacing $n\mathcal{Cat}$ by $n\mathcal{Alg}$. Theorem 7.2.4 gives a simple construction for the *free* n*-algebra* functor $(n-1)\mathcal{Alg}^+ \to n\mathcal{Alg}$, mapping an extended $(n-1)$-algebra (A, X) to an n-algebra $A[X]$. First, consider the $(n-1)$-globular bimodule associated to A, and extend it into an n-globular bimodule $A[X]$ as follows. The n-cells of $A[X]$ are the elements of the sum of the free A_0-bimodule over X and of a copy of A_{n-1}

$$\big(A_0 \otimes \mathbb{K}X \otimes A_0\big) \oplus A_{n-1},$$

quotiented by (7.4); the copy of A_{n-1} is used to define the identity map $A_{n-1} \to A[X]_n$. Thus, the n-cells of $A[X]$ are the linear combinations of elements axb, for a and b in A_0 and x in X, and of an $(n-1)$-cell c of A, considered up to (7.4), with source, target and identity maps given by

$$s(axb) = as(x)b, \quad s(c) = c, \quad t(axb) = at(x)b, \quad t(c) = c, \quad i(c) = c.$$

Now, define a product on cells of dimension above 0 as prescribed by (7.3), to equip $A[X]$ with a structure of n-globular algebra. Finally, define the compositions by (7.1) to obtain the n-algebra structure of $A[X]$.

The *quotient of* A *by* X is the n-algebra denoted by A/X, and obtained by quotient of the n-cells of A by the congruence $\approx_X$.

7.2.6. Polygraphs for associative algebras. For n a natural number, the category $n\mathcal{Pol}(\mathcal{Alg})$ of n*-polygraphs for associative algebras*, simply called n*-polygraphs* in this chapter, and the *free* n*-algebra functor* from $n\mathcal{Pol}(\mathcal{Alg})$ to $n\mathcal{Alg}$, are defined by induction on n as in §2.4.1. We start with $0\mathcal{Pol}(\mathcal{Alg}) = \mathcal{Set}$, and, for $n \geqslant 1$, we define $n\mathcal{Pol}(\mathcal{Alg})$ using a pullback similar to (2.1), where $(n-1)\mathcal{Alg}$ and $(n-1)\mathcal{Alg}^+$ replace $(n-1)\mathcal{Cat}$ and $(n-1)\mathcal{Cat}^+$. The category $\infty\mathcal{Pol}(\mathcal{Alg})$ of ∞*-polygraphs*, and the corresponding *free* ∞*-algebra functor*, mapping X to $\mathbb{K}\langle X\rangle$, are then obtained as limits. Thus, by construction, an ∞-polygraph X is a sequence $(X_0 \,|\, \cdots \,|\, X_n \,|\, \cdots)$ made of a set X_0 and, for every $n \geqslant 0$, an extension of the free n-algebra $\mathbb{K}\langle X_0\rangle[X_1]\cdots[X_n]$.

Fix $n \in \overline{\mathbb{N}}$, and let X be an n-polygraph. The *algebra presented by* X is the quotient algebra $\overline{X} = \mathbb{K}\langle X_0\rangle / X_1$. When the context is clear, we denote by $\overline{a}$ the image of a 0-cell a of $\mathbb{K}\langle X\rangle$ through the canonical projection. A *monomial of* $\mathbb{K}\langle X\rangle$ is an element of X_0^*. The monomials of $\mathbb{K}\langle X\rangle$ form a linear basis of the free algebra $\mathbb{K}\langle X_0\rangle$, and, if a is a 0-cell of $\mathbb{K}\langle X\rangle$, we define the *support of* a as the set $\mathrm{Supp}(a)$ of monomials that appear in its decomposition in this basis.

If $n \geqslant 1$, an $1 \leqslant k \leqslant n$ is a natural number, a k-*monomial of* $\mathbb{K}\langle X\rangle$ is a k-cell of $\mathbb{K}\langle X\rangle$ with shape $u\alpha v$, where α is a k-cell of X, and u and v are monomials of $\mathbb{K}\langle X\rangle$. By construction of $\mathbb{K}\langle X\rangle$, every k-cell a of $\mathbb{K}\langle X\rangle$ is a linear combination

$$a = \sum_{i=1}^{p} \lambda_i a_i + c \tag{7.5}$$

of pairwise distinct k-monomials $a_1, \ldots, a_p$ and of an identity k-cell c of $\mathbb{K}\langle X\rangle$, and this decomposition is unique up to the relation (7.4). If a is a k-cell of $\mathbb{K}\langle X\rangle$, the *size of* a is the minimum number of k-monomials of $\mathbb{K}\langle X\rangle$ required to write a as in (7.5), and we denote by $\mathrm{Cell}(a)$ the set of k-cells of X that appear in the corresponding k-monomials.

Let A be an algebra. A *presentation of* A is a 1-polygraph X such that A is isomorphic to $\overline{X}$. A *coherent presentation of* A is a 2-polygraph X such that A is isomorphic to $\overline{X}$ and X_2 is acyclic. And a *polygraphic resolution of* A is an ∞-polygraph X such that A is isomorphic to $\overline{X}$ and X_{n+1} is acyclic for every $n \geqslant 1$.

7.3. Convergent presentations of associative algebras

7.3.1. Rewriting steps and normal forms. Let X be a 1-polygraph. We say that X is *left-monomial* if, for every 1-cell α of X, the source of α is a monomial of $\mathbb{K}\langle X\rangle$ that does not belong to $\mathrm{Supp}(t(\alpha))$. In that case, the source of every 1-monomial of $\mathbb{K}\langle X\rangle$ is a monomial of $\mathbb{K}\langle X\rangle$.

Assume that X is left-monomial. A *rewriting step of* X is a 1-cell $\lambda f + 1_a$ of size 1 of the free 1-algebra $\mathbb{K}\langle X\rangle$ that satisfies the condition

$$\mathrm{Supp}(\lambda s(f) + a) = \{s(f)\} \sqcup \mathrm{Supp}(a),$$

i.e. such that $\lambda \neq 0$ and $s(f) \notin \mathrm{Supp}(a)$. A 1-cell of $\mathbb{K}\langle X\rangle$ is called *positive* if it is a 0-composite $f_1 \star_0 \cdots \star_0 f_p$ of rewriting steps of $\mathbb{K}\langle X\rangle$.

A 0-cell a of $\mathbb{K}\langle X\rangle$ is called *reduced* if $\mathbb{K}\langle X\rangle$ contains no rewriting step of source a, and a *normal form of* a is a reduced 0-cell b of $\mathbb{K}\langle X\rangle$ such that there exists a positive 1-cell $a \to b$ in $\mathbb{K}\langle X\rangle$. The reduced 0-cells of $\mathbb{K}\langle X\rangle$ form a linear subspace of $\mathbb{K}\langle X_0\rangle$ that is denoted by $\mathrm{Red}(X)$. Because X is left-monomial, the set of reduced monomials of $\mathbb{K}\langle X\rangle$, denoted by $\mathrm{Red}_m(X)$, forms a basis of $\mathrm{Red}(X)$. If a is a 0-cell of $\mathbb{K}\langle X\rangle$, a *normal form of* a is a reduced 0-cell b of $\mathbb{K}\langle X\rangle$ such that there exists a positive 1-cell $a \to b$ in $\mathbb{K}\langle X\rangle$.

7.3.2. Lemma. *Let X be a left-monomial 1-polygraph. Every 1-cell f of size 1 of $\mathbb{K}\langle X\rangle$ can be decomposed into $f = g \star_0 h^-$, where g and h are identities or rewriting steps of X.*

Proof. Write $f = \lambda f' + b$, where $f' : u \to a$ is a 1-monomial of $\mathbb{K}\langle X\rangle$. Let μ be the coefficient of u in b, so that $b = \mu u + c$ with c such that $u \notin \mathrm{Supp}(c)$. Put

$$g = (\lambda + \mu) f' + 1_c \qquad \text{and} \qquad h = \lambda a + \mu f' + 1_c.$$

The linearity of $\star_0$ gives $f = g \star_0 h^-$. Since u does not belong to $\mathrm{Supp}(a)$ or $\mathrm{Supp}(c)$, each of the 1-cells g and h is either an identity (if $\lambda + \mu = 0$ for g, if $\mu = 0$ for h) or a rewriting step. $\square$

7.3.3. Termination. Assume that X is a set and that $\vdash$ is a binary relation on the free monoid X^*. We say that $\vdash$ is *stable by product* if $u \vdash u'$ implies $vuw \vdash vu'w$ for all u, u', v and w in X^*. If Y is a left-monomial extension of $\mathbb{K}\langle X\rangle$, we say that $\vdash$ is *compatible with* Y if $u \vdash v$ holds for every 1-cell $y : u \to a$ of Y and every monomial $v \in \mathrm{Supp}(a)$. The relation $\vdash$ is extended to the 0-cells of the free algebra $\mathbb{K}\langle X\rangle$ by putting $a \vdash b$ if the following hold:

(i) $\mathrm{Supp}(a) \setminus \mathrm{Supp}(b) \neq \emptyset$,

(ii) for every v in $\mathrm{Supp}(b) \setminus \mathrm{Supp}(a)$, there exists u in $\mathrm{Supp}(a) \setminus \mathrm{Supp}(b)$, such that $u \vdash v$.

As a consequence, if u is a monomial and a is a 0-cell of $\mathbb{K}\langle X\rangle$, then $u \vdash a$ holds if, and only if, $u \vdash v$ holds for every $v \in \mathrm{Supp}(a)$. The relation $\vdash$ on the 0-cells of $\mathbb{K}\langle X\rangle$ corresponds to the restriction to finite subsets of X^* of the so-called multiset relation generated by $\vdash$. See [10, §2.5] for the general definition, and a proof that $\vdash$ is wellfounded on 0-cells if, and only if, it is so on monomials.

Let X be a left-monomial 1-polygraph. Define $\succcurlyeq_X$ as the smallest reflexive-transitive binary relation on X_0^* that is stable by product and compatible with X_1. When the context is clear, we simply write $\succcurlyeq$ for $\succcurlyeq_X$. We say that X *terminates* if the relation $\succcurlyeq_X$ is wellfounded, i.e. if $\succcurlyeq_X$ is a wellfounded order, called the *rewrite order of* X.

7.3.4. Branchings and confluence. Assume that X is a left-monomial 1-polygraph. Branchings and local branchings are defined as in §2.5.3. For a branching (f, g) of X of source a, put

$$\lambda u(f, g)v + b = (\lambda ufv + b,\ \lambda ugv + b).$$

A local branching is *trivial* if it is of one of the following three shapes:

(i) $\lambda(f, f) + b$, for a 1-monomial $f : u \to a$ of $\mathbb{K}\langle X\rangle$, a nonzero scalar λ, and a 0-cell b of $\mathbb{K}\langle X\rangle$, with $u \notin \mathrm{Supp}(b)$.

(ii) $(\lambda f + \mu v + c,\ \lambda u + \mu g + c)$, for 1-monomials $f : u \to a$ and $g : v \to b$ of $\mathbb{K}\langle X\rangle$, nonzero scalars λ and μ, and a 0-cell c of $\mathbb{K}\langle X\rangle$, with $u \neq v$ and $u, v \notin \mathrm{Supp}(c)$.

(iii) $\lambda(fv, ug) + c$, for 1-monomials $f : u \to a$ and $g : v \to b$ of $\mathbb{K}\langle X\rangle$, a nonzero scalar λ, and a 0-cell c of $\mathbb{K}\langle X\rangle$, with $uv \notin \mathrm{Supp}(c)$.

After examination, nontrivial local branchings are of the form $\lambda(f, g) + c$, for 1-monomials $f : u \to a$ and $g : u \to b$ of $\mathbb{K}\langle X\rangle$ such that (f, g) is nontrivial with monomial source u, for λ a nonzero scalar, and c a 0-cell of $\mathbb{K}\langle X\rangle$, with $u \notin \mathrm{Supp}(c)$. Such a nontrivial local branching is *critical* if $\lambda = 1$ and $c = 0$, and if it cannot be factored $(f, g) = u(f', g')v$ in a nontrivial way. Note that every nontrivial local branching has a unique decomposition $\lambda u(f_0, g_0)v + c$, with (f_0, g_0) critical.

Confluence, local confluence and critical confluence are then defined as in §2.5.3.

7.3.5. Convergence. Let X be a left-monomial 1-polygraph. We say that X is *convergent* if it is both terminating and confluent. In that case, every 0-cell a of $\mathbb{K}\langle X\rangle$ has a unique normal form, denoted by $\widehat{a}$, such that $\overline{a} = \overline{b}$ holds in $\overline{X}$ if, and only if, $\widehat{a} = \widehat{b}$ holds in $\mathbb{K}\langle X\rangle$. Hence, if X is a convergent presentation of an algebra A, the assignment of each element a of A to the normal form of any representative of a in $\mathbb{K}\langle X\rangle$, written $\widehat{a}$ by extension, defines a section $A \rightarrowtail \mathbb{K}\langle X\rangle$ of the canonical projection, where A is seen as a 1-algebra with identity 1-cells only. The section is linear (the normal form of $\lambda a + \mu b$ is $\lambda\widehat{a} + \mu\widehat{b}$), it preserves the unit (termination implies $\widehat{1} = 1$), but $\widehat{ab} \neq \widehat{a}\widehat{b}$ in general. The completion procedure, developed by Buchberger for commutative algebras [42] and by Knuth and Bendix for term rewriting systems [126], adapts to terminating left-monomial 1-polygraphs in a straightforward way.

7.3.6. Theorem. *Let* A *be an algebra and* X *be a convergent presentation of* A*. Then the set* $\mathrm{Red}_m(X)$ *of reduced monomials of* $\mathbb{K}\langle X\rangle$ *is a linear basis of* A*. As a consequence, the vector space* $\mathrm{Red}(X)$*, equipped with the product defined by* $a \cdot b = \widehat{ab}$*, is an algebra isomorphic to* A*.*

7.3.7. Example. Let A be the algebra presented by the 1-polygraph $X = \langle x, y \mid \alpha : xy \to x^2 \rangle$, which terminates, because $xy > x^2$ holds for the deglex order generated by $y > x$. This presentation is also confluent, because it has no critical branching (see Theorem 7.5.2). Hence, the set $\mathrm{Red}_m(X) = \{ y^i x^j \mid i, j \in \mathbb{N} \}$ is a linear basis of the algebra A. Moreover, the product defined by

$$y^i x^j \cdot y^k x^l \;=\; \begin{cases} y^i x^{j+k+l} & \text{if } j \geqslant k \\ y^{i-j+k} x^{2j+l} & \text{if } j \leqslant k \end{cases}$$

turns $\mathrm{Red}(X)$ into an algebra that is isomorphic to A.

7.4. Comparison with Gröbner bases and Poincaré-Birkhoff-Witt bases

In this section, if X is a 1-polygraph, we denote by $I(X)$ the ideal of $\mathbb{K}\langle X_0\rangle$ generated by the boundaries $\partial(a) = s(a) - t(a)$ of the 1-cells of X, so that $\overline{X} \simeq \mathbb{K}\langle X_0\rangle / I(X)$.

7.4.1. Lemma. *Let* X *be a* 1*-polygraph. For all* 0*-cells* a *and* b *of* $\mathbb{K}\langle X\rangle$*, the* 0*-cell* $a - b$ *belongs to the ideal* $I(X)$ *if, and only if, there exists a* 1*-cell* $f : a \to b$ *in* $\mathbb{K}\langle X\rangle$*. As a consequence,* $I(X)$ *exactly contains the* 0*-cells* a *of* $\mathbb{K}\langle X\rangle$ *such that* $\overline{a} = 0$ *holds in* $\overline{X}$*.*

Proof. Assume $a - b \in I(X)$, i.e. $a - b = \sum_{1 \leqslant i \leqslant p} \lambda_i u_i \partial(\alpha_i) v_i$. Then

$$f \;=\; \sum_{i=1}^{p} \lambda_i u_i \alpha_i v_i \;+\; \big(a - \sum_{i=1}^{p} \lambda_i u_i s(\alpha_i) v_i\big).$$

defines a 1-cell $f : a \to b$ of $\mathbb{K}\langle X\rangle$. Conversely, let $f : a \to b$ be a 1-cell of $\mathbb{K}\langle X\rangle$. Write $f = f_1 \star_0 \cdots \star_0 f_p$, with $f_i = \lambda_i u_i \alpha_i v_i + h_i$ for every $1 \leqslant i \leqslant p$. Since $t(f_i) = s(f_{i+1})$, we have $a - b = \partial(f_1) + \cdots + \partial(f_p)$. Moreover, $\partial(f_i) = \lambda_i u_i \partial(\alpha_i) v_i$ implies that each $\partial(f_i)$ belongs to $I(X)$, and thus so does $a - b$. Finally, if one applies the equivalence to the case $b = 0$, since $\overline{0} = 0$ holds in $\overline{X}$, we get that a is in $I(X)$ if, and only if, we have $\overline{a} = 0$ in $\overline{X}$. □

7.4.2. Proposition. *Let* X *be a terminating left-monomial* 1*-polygraph. Then, as a vector space,* $\mathbb{K}\langle X_0\rangle$ *admits the decomposition*

$$\mathbb{K}\langle X_0\rangle = \mathrm{Red}(X) + I(X), \tag{7.6}$$

and the following assertions are equivalent:

(i) *X is confluent.*

(ii) *Every* 0*-cell of* $I(X)$ *admits* 0 *as a normal form.*

(iii) *As a vector space,* $\mathbb{K}\langle X_0\rangle$ *admits the direct decomposition* $\mathbb{K}\langle X_0\rangle = \mathrm{Red}(X) \oplus I(X)$*.*

Proof. Since X terminates, every 0-cell a of $\mathbb{K}\langle X\rangle$ admits at least a normal form b, i.e. a reduced 0-cell b such that there exists a positive 1-cell $f : a \to b$ in $\mathbb{K}\langle X\rangle$. We obtain (7.6) by writing $a = b + (a - b)$, and by observing that b belongs to $\mathrm{Red}(X)$, by hypothesis, and that $a - b$ is in $I(X)$, by Lemma 7.4.1.

(i) $\Rightarrow$ **(ii)**. By Lemma 7.4.1, if a is in $I(X)$, then there exists a 1-cell $f : a \to 0$ in $\mathbb{K}\langle X\rangle$. Since X is confluent, a and 0 have the same normal form, if any. And, since 0 is reduced, 0 is a normal form of a.

(ii) $\Rightarrow$ **(iii)**. Using (7.6), it is sufficient to prove that $\mathrm{Red}(X) \cap I(X)$ is reduced to 0. On the one hand, if a is in $\mathrm{Red}(X)$, then a is reduced and, thus, admits itself as only normal form. On the other hand, if a is in $I(X)$, then a admits 0 as a normal form by hypothesis.

(iii) $\Rightarrow$ **(i)**. Consider a branching (f, g) of X, with $f : a \to b$ and $g : a \to c$. Since X terminates, each of b and c admits at least one normal form, say b' and c' respectively. Hence, there exist positive 1-cells $h : b \to b'$ and $k : c \to c'$ in $\mathbb{K}\langle X\rangle$. Note that the difference $b' - c'$ is also reduced. Moreover, the 1-cell $(f \star_0 h)^- \star_0 (g \star_0 k)$ has b' as source and c' as target. This implies, by Lemma 7.4.1, that $b' - c'$ also belongs to $I(X)$. The hypothesis gives $b' - c' = 0$, so that (f, g) is confluent. □

7.4.3. Monomial orders and Gröbner bases. Let X be a set. A wellfounded total order on X^*, whose strict part is stable by product, is called a *monomial order on* $\mathbb{K}\langle X\rangle$. If X is a left-monomial 1-polygraph, and $\preccurlyeq$ is a monomial order on $\mathbb{K}\langle X_0\rangle$ that is compatible with X_1, then X terminates: the order $\succcurlyeq$ is wellfounded, and $a \succ_X b$ implies $a \succ b$ for all 0-cells a and b. However, the converse implication does not hold, as illustrated by Example 7.4.7.

Let X be a set and $\preccurlyeq$ be a monomial order on the free algebra $\mathbb{K}\langle X\rangle$. If a is a nonzero element of $\mathbb{K}\langle X\rangle$, the *leading monomial of* a is the maximum element $\mathrm{lm}(a)$ of $\mathrm{Supp}(a)$ for $\preccurlyeq$ (or 0 if $\mathrm{Supp}(a)$ is empty), the *leading coefficient of* a is the coefficient $\mathrm{lc}(a)$ of $\mathrm{lm}(a)$ in a, and the *leading term of* a is the element $\mathrm{lt}(a) = \mathrm{lc}(a)\,\mathrm{lm}(a)$ of $\mathbb{K}\langle X\rangle$. Let I be an ideal of $\mathbb{K}\langle X\rangle$. A *Gröbner basis for* $(I, \preccurlyeq)$ is a subset $\mathcal{G}$ of I such that the ideals of $\mathbb{K}\langle X\rangle$ generated by $\mathrm{lm}(I)$ and by $\mathrm{lm}(\mathcal{G})$ coincide.

7.4.4. Proposition. *If X is a convergent left-monomial 1-polygraph, and $\preccurlyeq$ is a monomial order on $\mathbb{K}\langle X_0\rangle$ that is compatible with X_1, then $\partial(X_1)$ is a Gröbner basis for $(I(X), \preccurlyeq)$.*

Conversely, let X be a set, let $\preccurlyeq$ be a monomial order on $\mathbb{K}\langle X\rangle$, let I be an ideal of $\mathbb{K}\langle X\rangle$ and $\mathcal{G}$ be a subset of I. Define $\mathcal{G}^\sharp$ as the 1-polygraph with 0-cells X and one 1-cell $\mathrm{lm}(a) \to \mathrm{lm}(a) - \frac{1}{\mathrm{lc}(a)}a$ for each a in $\mathcal{G}$. If $\mathcal{G}$ is a Gröbner basis for $(I, \preccurlyeq)$, then $\mathcal{G}^\sharp$ is a convergent left-monomial presentation of $\mathbb{K}\langle X\rangle/I$, such that $I(\mathcal{G}^\sharp) = I$, and $\preccurlyeq$ is compatible with $\mathcal{G}^\sharp_1$.

Proof. If X is convergent, then $\partial(\alpha)$ is in $I(X)$ for every 1-cell α of X. Since $\preccurlyeq$ is compatible with X_1, we have $\mathrm{lm}(\partial(\alpha)) = s(\alpha)$ for every 1-cell α of X. Now, if a is in $I(X)$, it is a linear combination $a = \sum_i \lambda_i u_i \partial(\alpha_i) v_i$, where α_i is a 1-cell of X, and u_i and v_i are monomials of $\mathbb{K}\langle X\rangle$. This implies $\mathrm{lm}(a) = u_i s(\alpha_i) v_i = u_i\,\mathrm{lm}(\partial(\alpha_i)) v_i$ for some i. Thus $\partial(X_1)$ is a Gröbner basis for $(I(X), \preccurlyeq)$.

Conversely, assume that $\mathcal{G}$ is a Gröbner basis for $(I, \preccurlyeq)$. By definition, $\preccurlyeq$ is compatible with $\mathcal{G}^\sharp_1$, hence $\mathcal{G}^\sharp_1$ terminates, and $I(\mathcal{G}^\sharp) = I$ holds, so that $\overline{\mathcal{G}^\sharp} \simeq \mathbb{K}\langle X\rangle/I$. Moreover, the reduced monomials of $\mathbb{K}\langle \mathcal{G}^\sharp\rangle$ are the monomials of $\mathbb{K}\langle X\rangle$ that cannot be decomposed as $u\,\mathrm{lm}(a)v$ with a

in $\mathcal{G}$, and u and v monomials of $\mathbb{K}\langle X\rangle$. Thus, if a reduced 0-cell a of $\mathbb{K}\langle \mathcal{G}^\sharp\rangle$ is in I, then $\mathrm{lm}(a) = 0$, because $\mathcal{G}$ is a Gröbner basis of $(I, \preccurlyeq)$. As a consequence of Proposition 7.4.2, we get that $\mathcal{G}^\sharp$ is confluent. □

7.4.5. Poincaré-Birkhoff-Witt bases. Let A be an N-homogeneous algebra, for $N \geqslant 2$, let X be a generating set of A, concentrated in degree 1, and let $\preccurlyeq$ be a monomial order on $\mathbb{K}\langle X\rangle$. A *Poincaré-Birkhoff-Witt* (*PBW*) *basis for* $(A, X, \preccurlyeq)$ is a subset $\mathcal{B}$ of X^* such that:

- **(i)** $\mathcal{B}$ is a linear basis of A, with $[u]_{\mathcal{B}}$ denoting the decomposition of $u \in X^*$ in the basis $\mathcal{B}$,
- **(ii)** for all u and v in $\mathcal{B}$, we have $uv \succcurlyeq [uv]_{\mathcal{B}}$,
- **(iii)** an element u of X^* belongs to $\mathcal{B}$ if, and only if, for every decomposition $u = vu'w$ of u in X^* such that u' has degree N, then u' is in $\mathcal{B}$.

7.4.6. Proposition. *If X is a convergent left-monomial N-homogeneous presentation of an algebra A, and $\preccurlyeq$ is a monomial order on $\mathbb{K}\langle X_0\rangle$ that is compatible with X_1, then the set* $\mathrm{Red}_m(X)$ *of reduced monomials of* $\mathbb{K}\langle X\rangle$ *is a PBW basis for* $(A, X_0, \preccurlyeq)$.

Conversely, let A be an N-homogeneous algebra, let X be a generating set of A that is concentrated in degree 1, let $\preccurlyeq$ a monomial order on $\mathbb{K}\langle X\rangle$, and $\mathcal{B}$ be a PBW basis of $(A, X, \preccurlyeq)$. *Define $\mathcal{B}^\sharp$ as the 1-polygraph with 0-cells X and with one 1-cell $uv \to [uv]_{\mathcal{B}}$ for all u and v in $\mathcal{B}$ such that uv has degree N and $uv \neq [uv]_{\mathcal{B}}$. Then $\mathcal{B}^\sharp$ is a convergent left-monomial N-homogeneous presentation of A, such that* $\mathrm{Red}_m(\mathcal{B}^\sharp) = \mathcal{B}$, *and $\preccurlyeq$ is compatible with $\mathcal{B}^\sharp_1$.*

Proof. If X is a convergent left-monomial presentation of A, Theorem 7.3.6 implies that the set $\mathrm{Red}_m(X)$ of reduced monomials of $\mathbb{K}\langle X\rangle$ is a linear basis of A. The fact that $\preccurlyeq$ is compatible with X_1 implies $uv \succcurlyeq [uv]_{\mathcal{B}}$, and the third axiom of the definition of a PBW basis comes from the definition of a reduced monomial for an N-homogeneous left-monomial 1-polygraph.

Conversely, assume that $\mathcal{B}$ is a PBW basis for $(A, X, \preccurlyeq)$. By definition, $\mathcal{B}^\sharp$ is N-homogeneous and left-monomial, the third axiom of the definition of a PBW basis implies $\mathrm{Red}_m(\mathcal{B}^\sharp) = \mathcal{B}$, and the second axiom gives termination of $\mathcal{B}^\sharp$. By Proposition 7.4.2, it is sufficient to prove that $\mathrm{Red}(\mathcal{B}^\sharp) \cap I(\mathcal{B}^\sharp) = 0$ to get confluence: on the one hand, a reduced 0-cell a of $\mathrm{Red}(\mathcal{B}^\sharp)$ is a linear combination of 0-cells of $\mathcal{B}$, so that a is its only normal form; and, on the other hand, if a belongs to $I(\mathcal{B}^\sharp)$, then a admits 0 as a normal form by Lemma 7.4.1. Finally, $\overline{\mathcal{B}^\sharp}$ is isomorphic to $\mathrm{Red}(\mathcal{B}^\sharp)$, i.e. to $\mathbb{K}\mathcal{B}$, hence to A, by Theorem 7.3.6 and because $\mathcal{B}$ is a linear basis of A. □

7.4.7. Example. Convergent presentations of associative algebras are strict generalisations of Gröbner bases and PBW bases, because rewrite orders generated by terminating polygraphs strictly contain monomial orders that are compatible with 1-cells. For example, let us prove that the 1-polygraph $X = \big(x, y, z \,\big|\, xyz \to x^3 + y^3 + z^3\big)$ terminates. For every monomial u of $\mathbb{K}\langle X\rangle$, denote by $A(u)$ the number of factors xyz that occur in u, by $B(u)$ the number of y that u contains, and put $\Phi(u) = 3A(u) + B(u)$. It is sufficient to check that $\Phi(uxyzv)$ is strictly greater than each of $\Phi(ux^3v)$, $\Phi(uy^3v)$ and $\Phi(uz^3v)$, for all monomials u and v of $\mathbb{K}\langle X\rangle$:

$$\Phi(uxyzv) = \Phi(u) + \Phi(v) + 4, \qquad \Phi(ux^3v) = \begin{cases} \Phi(u) + \Phi(v) + 3 & \text{if } v = yzv', \\ \Phi(u) + \Phi(v) & \text{otherwise,} \end{cases}$$

$$\Phi(uy^3v) = \Phi(u) + \Phi(v) + 3, \qquad \Phi(uz^3v) = \begin{cases} \Phi(u) + \Phi(v) + 3 & \text{if } u = u'xy, \\ \Phi(u) + \Phi(v) & \text{otherwise.} \end{cases}$$

However, no monomial order on $\mathbb{K}\langle X_0\rangle$ is compatible with X_1, because, for such an order $\succ$, one of the monomials x^3, y^3, z^3 is always greater than xyz. Indeed, since $\succ$ is total, one of x, y or z is greater than the other two. If it is x, the case of z being symmetric, $x \succ y$ implies $x^2 \succ yx$ and $x \succ z$ implies $yx \succ yz$, so that $x^2 \succ yz$, hence $x^3 \succ xyz$. Now, if $y \succ x$ and $y \succ z$, we get $y^2 \succ xy$, thus $y^2z \succ xyz$ and $y^3 \succ y^2z$.

7.5. Coherent presentations of associative algebras

If X is a left-monomial 1-polygraph, and Y is an extension of $\mathbb{K}\langle X\rangle$, we define the notions of (*local*, *critical*) Y-*confluence* and Y-*convergence* for X as in §3.2.1.

7.5.1. Lemma. *Let X be a left-monomial 1-polygraph, and Y be an extension of $\mathbb{K}\langle X\rangle$, such that X is Y-confluent at every 0-cell $b \prec a$ for some fixed 0-cell a of $\mathbb{K}\langle X\rangle$. Let f be a 1-cell of $\mathbb{K}\langle X\rangle$ that admits a decomposition*

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_p} a_p$$

into 1-cells of size 1 such that $a_i \prec a$ holds for every $0 < i < p$. Then there exist positive 1-cells g and h in $\mathbb{K}\langle X\rangle$ and a 2-cell F in $\mathbb{K}\langle X\rangle[Y]$ as in

$$\begin{tikzcd}
& a_p \arrow[dr, bend left, "h"] \arrow[d, phantom, "\Downarrow F"] & \\
a_0 \arrow[ur, bend left, "f"] \arrow[rr, bend right, "g"'] & {} & a'.
\end{tikzcd}$$

Proof. Proceed by induction on $p \geqslant 0$. If $p = 0$, then $f = 1_{a_0}$, so taking $g = h = 1_{a_0}$ and $F = 1_f$ proves the result. Otherwise, let us construct

$$\begin{tikzcd}
& & a_p \arrow[dr, bend left, "h_2"] \arrow[d, phantom, "\Downarrow F"] & & \\
& a_1 \arrow[ur, bend left, "f_2 \star_0 \cdots \star_0 f_p"] \arrow[rr, "g_2"] \arrow[dr, "h_1"] \arrow[d, phantom, "="] & {} & b_2 \arrow[dr, bend left, "k_2"] \arrow[d, phantom, "\Downarrow G"] & \\
a_0 \arrow[ur, bend left, "f_1"] \arrow[rr, bend right, "g_1"'] & {} & b_1 \arrow[rr, bend right, "k_1"'] & {} & a'
\end{tikzcd}$$

First, apply Lemma 7.3.2 to the 1-cell f_1 of size 1 to get g_1 and h_1 such that $f_1 = g_1 \star_0 h_1^-$. Then, apply the induction hypothesis to obtain g_2, h_2 and F. Finally, apply the Y-confluence hypothesis to (h_1, g_2) to produce k_1, k_2 and G. □

7.5.2. Theorem. *Let X be a terminating left-monomial 1-polygraph, and Y be an extension of $\mathbb{K}\langle X\rangle$. If X is critically Y-confluent then it is Y-confluent.*

Proof. We proceed by noetherian induction on the source of the branchings to prove that Y-critical confluence implies Y-local confluence, which in turn implies Y-confluence. The proof of the second implication is the same as for Proposition 3.2.2. The proof of the first implication is based on a case analysis on the type of local branching, as in Proposition 3.2.3, but the argument differs in that it also requires the termination hypothesis, see [95, Remark 4.2.4]. Here, we describe the case of a nontrivial local branching, and refer to [95, Theorem 4.2.1] for the trivial case, whose study follows a similar pattern. For an overlapping branching $(\lambda f + c, \lambda g + c)$, let us construct

$$\begin{tikzcd}
 & \lambda a + c \arrow[rr, bend left, "f_1'"] \arrow[dr, dotted, "\lambda f' + c"'] & \arrow[d, phantom, "\Downarrow G"] & a' \arrow[dr, bend left, "f_2'"] & \\
\lambda u + c \arrow[ur, bend left, "\lambda f + c"] \arrow[dr, bend right, "\lambda g + c"'] & \arrow[l, phantom, "\Downarrow F"] & \lambda e + c \arrow[ur, "h"] \arrow[dr, "k"'] & \arrow[r, phantom, "\Downarrow I"] & d \\
 & \lambda b + c \arrow[rr, bend right, "g_1'"'] \arrow[ur, dotted, "\lambda g' + c"] & \arrow[u, phantom, "\Downarrow H"] & b' \arrow[ur, bend right, "g_2'"'] &
\end{tikzcd}$$

Consider the unique decomposition $(f, g) = v(f_0, g_0)w$, with (f_0, g_0) critical. Since (f_0, g_0) is Y-confluent by hypothesis, we obtain

$$\begin{tikzcd}
 & a_0 \arrow[dr, bend left, "f_0'"] \arrow[d, phantom, "\Downarrow F_0"] & \\
u_0 \arrow[ur, bend left, "f_0"] \arrow[dr, bend right, "g_0"'] & {} & e_0 \\
 & b_0 \arrow[ur, bend right, "g_0'"'] &
\end{tikzcd}$$

Define the positive 1-cells $f' = vf_0'w$ and $g' = vg_0'w$, and the 2-cell $F = vF_0w$. The dotted 1-cells are not positive in general, for example if $\mathrm{Supp}(c)$ intersects $\mathrm{Supp}(a)$ or $\mathrm{Supp}(b)$. However, the 1-cell f' is positive, so that it is a 0-composite $f' = l_1 \star_0 \cdots \star_0 l_p$ of rewriting steps. As a consequence, we have the chain of inequalities $u \succ s(l_1) \succ \cdots \succ s(l_p) \succ e$. Since we have $\lambda \neq 0$ and $u \notin \mathrm{Supp}(c)$ by hypothesis, the inequality $\lambda u + c \succ \lambda s(l_i) + c$ holds for every i, so that the $\lambda f' + c = (\lambda l_1 + c) \star_1 \cdots \star_1 (\lambda l_p + c)$ satisfies the hypotheses of Lemma 7.5.1. This gives f_1', h and G. We proceed similarly with the 1-cell $\lambda g' + c$ to obtain g_1', k and H. Finally, we apply the induction hypothesis on (h, k), since $\lambda u + c \succ \lambda e + c$, to get f_2', g_2' and I. □

7.5.3. Theorem (Squier's theorem for algebras). *Let* X *be a convergent left-monomial* 1-*polygraph, and* Y *be an extension of* $\mathbb{K}\langle X\rangle$. *If* X *is* Y-*convergent, then* Y *is acyclic. In particular, if* Y *contains a* 2-*cell*

$$\begin{tikzcd}
 & b \arrow[dr, bend left, "h"] \arrow[d, phantom, "\Downarrow F"] & \\
a \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & {} & d \\
 & c \arrow[ur, bend right, "k"'] &
\end{tikzcd}$$

for every critical branching (f, g) *of* X, *with* h *and* k *positive* 1-*cells of* $\mathbb{K}\langle X\rangle$, *then* Y *is acyclic.*

Proof. The proof of the first assertion is almost identical to the one of Proposition 3.2.4. The only difference is that, here, we have to invoke Lemma 7.3.2, to factorise a 1-cell f of $\mathbb{K}\langle X\rangle$ into a composite $f = g_1 \star_0 h_1^- \star_0 \cdots \star_0 g_p \star_0 h_p^-$ where each g_i and h_i is positive. □

7.5.4. Example. From [176, §4.3], we consider the algebra

$$A = \langle x, y, z \mid x^2 + yz = 0,\ x^2 + \lambda zy = 0 \rangle\,,$$

where $\lambda \neq 0, 1$ is a fixed scalar. The algebra A admits the presentation $X = (x, y, z \mid \alpha, \beta)$, with $\alpha : yz \to -x^2$ and $\beta : zy \to -\mu x^2$, and where $\mu = \lambda^{-1}$. The deglex order generated by $z > y > x$ satisfies $yz > x^2$ and $zy > x^2$, proving that X terminates. However, X is not confluent. Indeed, it has two critical branchings

$$-x^2y \xleftarrow{\ \alpha y\ } yzy \xrightarrow{\ y\beta\ } -\mu y x^2 \qquad \text{and} \qquad -\mu x^2 z \xleftarrow{\ \beta z\ } zyz \xrightarrow{\ z\alpha\ } -zx^2$$

and neither of them is confluent, because the monomials x^2y, yx^2, x^2z and zx^2 are reduced. The adjunction of $\gamma : yx^2 \to \lambda x^2 y$ and $\delta : zx^2 \to \delta\mu x^2 z$ gives a left-monomial 1-polygraph Y that also presents A, and that also terminates, because of $yx^2 > x^2y$ and $zx^2 > x^2z$. Moreover, each one of the four critical branchings of Y is confluent:

$$\begin{tikzcd}
& -x^2y \\
yzy \arrow[ur, bend left, "\alpha y"] \arrow[dr, bend right, "y\beta"'] \arrow[r, phantom, "\Downarrow F"] & \\
& -\mu y x^2 \arrow[uu, bend right, "-\mu\gamma"']
\end{tikzcd}
\quad
\begin{tikzcd}
& -\mu x^2 z \\
zyz \arrow[ur, bend left, "\beta z"] \arrow[dr, bend right, "z\alpha"'] \arrow[r, phantom, "\Downarrow G"] & \\
& -zx^2 \arrow[uu, bend right, "-\delta"']
\end{tikzcd}
\quad
\begin{tikzcd}
& -x^4 & \\
yzx^2 \arrow[ur, "\alpha x^2"] \arrow[dr, "y\delta"'] \arrow[rr, phantom, "\Downarrow H"] & & x^2yz \arrow[ul, "x^2\alpha"'] \\
& \mu y x^2 z \arrow[ur, "\mu\gamma z"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& -\mu x^4 & \\
zyx^2 \arrow[ur, "\beta x^2"] \arrow[dr, "z\gamma"'] \arrow[rr, phantom, "\Downarrow I"] & & x^2zy \arrow[ul, "x^2\beta"'] \\
& \lambda z x^2 y \arrow[ur, "\lambda\delta y"'] &
\end{tikzcd}$$

Theorem 7.5.3 says that $\langle Y \mid F, G, H, I \rangle$ is a coherent presentation of A. Following the same steps as in Example 5.4.5, this coherent presentation can be reduced to $(X \mid \emptyset)$ by eliminating H and I after examination of the critical 3-branchings of Y, and, then, eliminating F with γ and G with δ.

7.6. Polygraphic resolutions and Koszulness

7.6.1. Collapsing schemes. Let X be an ∞-polygraph, and Y be a graded subset of X. A *collapsing scheme of* X *onto* Y is an injective graded partial map $\varphi : X \to X$ of degree 1 such that:

(i) as a graded set, X admits the three-block partition $X = \mathrm{im}(\varphi) \sqcup Y \sqcup \mathrm{dom}(\varphi)$,

(ii) for every x in $\mathrm{dom}(\varphi)$, the boundary of φ_x satisfies $\partial(\varphi_x) = \lambda x + a$, where λ is a nonzero scalar, and a is an n-cell of $\mathbb{K}\langle X \rangle$ such that x does not belong to the set $\mathrm{Cell}(a)$ of n-cells of X that appear in a,

(iii) putting $x \trianglerighteq_\varphi y$, for all x in $\mathrm{dom}(\varphi)$ and y in $\mathrm{Cell}(\partial(\varphi_x))$, defines a wellfounded order on the cells of X.

If φ is a collapsing scheme of X onto Y, and $x \in \mathrm{dom}(\varphi)$,with $\partial(\varphi_x) = \lambda x + a$, we put $\widetilde{x} = -\frac{1}{\lambda} a$.

We refer to [95, §5.3] for the proof of the following result, and to Examples 7.6.8 and 7.6.9 for illustrations:

7.6.2. Theorem. *Let* X *be an* ∞*-polygraph,* Y *be a graded subset of* X*, and* φ *be a collapsing scheme of* X *onto* Y*. Setting, for every* $n \geqslant 1$ *and every* n*-cell* x *of* Y*,*

$$\overline{s}(x) = \pi(s(x)) \qquad \textit{and} \qquad \overline{t}(x) = \pi(t(x)),$$

and, for every $n \geqslant 0$ *and every* n*-cell* x *of* X,

$$\pi(x) = \begin{cases} x & \text{if } x \in Y, \\ \pi(\widetilde{x}) & \text{if } x \in \mathrm{dom}(\varphi), \\ 1_{\overline{s}(x)} & \text{if } x \in \mathrm{im}(\varphi), \end{cases} \tag{7.7}$$

defines, by mutual induction, a structure of ∞*-polygraph on* Y *and a morphism* $\pi : \mathbb{K}\langle X\rangle \to \mathbb{K}\langle Y\rangle$ *of* ∞*-algebras. Moreover, if* X *is a polygraphic resolution of an algebra* A*, then so is* Y.

7.6.3. The standard polygraphic resolution. Let $A = \mathbb{K} \oplus A_+$ be an augmented algebra. For $n \geqslant 1$, define the vector space

$$A^{(n)} = \bigoplus_{\substack{1 \leqslant k \leqslant n \\ i_1 + \cdots + i_k = n}} A_+^{\otimes i_1 + 1} \otimes \cdots \otimes A_+^{\otimes i_k + 1}.$$

Vertical bars are used to denote the innermost products of copies of A_+, so that $A^{(n)}$ is made of the linear combinations of elements $a_0^1 | \cdots | a_{i_1}^1 \otimes \cdots \otimes a_0^k | \cdots | a_{i_k}^k$ with n vertical bars. For $1 \leqslant i \leqslant n$, let d_i^- (resp. d_i^+) be the linear map from $A^{(n)}$ to $A^{(n-1)}$ that replaces the i^{th} vertical bar, counting from the left, with a tensor (resp. the product of A_+). For example, we have $d_2^-(a \otimes b|c \otimes d|e) = a \otimes b|c \otimes d \otimes e$ and $d_1^+(a \otimes b|c \otimes d|e) = a \otimes bc \otimes d|e$. Define $d_i^{(j)}$ as d_i^- if j is odd, and as d_i^+ if j is even.

Assume that $\mathcal{B}$ is a linear basis of A that contains 1, and put $\mathcal{B}_+ = \mathcal{B} \setminus \{1\}$. Then, setting $\mathrm{Std}(\mathcal{B})_n = \mathcal{B}_+^{n+1}$ and, for every $\underline{u}$ in $\mathcal{B}_+^{n+1}$,

$$s(\underline{u}) = \sum_{\substack{1 \leqslant k \leqslant n \\ 1 \leqslant i_1 < \cdots < i_k \leqslant n}} (-1)^{k+1} d_{i_1}^{(n+1-i_1)} \cdots d_{i_k}^{(n+1-i_k)}(\underline{u}),$$

$$t(\underline{u}) = \sum_{\substack{1 \leqslant k \leqslant n \\ 1 \leqslant i_1 < \cdots < i_k \leqslant n}} (-1)^{k+1} d_{i_1}^{(n-i_1)} \cdots d_{i_k}^{(n-i_k)}(\underline{u}),$$

defines a polygraphic resolution of A, called the *standard polygraphic resolution of* A. The complete proof, see [95, Theorem 6.1.2], relies on the construction of the linear analogue of the contractions of §4.2.4, see [95, §5.2], given by $\sigma_{u_0|\cdots|u_n \otimes u_{n+1}} = u_0 | \cdots | u_{n+1}$.

Assume that X is a convergent presentation of A. By Theorem 7.3.6, the set $\mathrm{Red}_m(X)$ of reduced monomials of $\mathbb{K}\langle X\rangle$ is a linear basis of A, so we obtain a polygraphic resolution $\mathrm{Std}(\mathrm{Red}_m(X))$. An n-cell $u_0 | \cdots | u_n$ of $\mathrm{Std}(\mathrm{Red}_m(X))$ is called *critical* if

- **(i)** u_0 is a 1-cell of X,
- **(ii)** for every $1 \leqslant i \leqslant n$, the 1-cell $u_{i-1}u_i$ is not reduced,
- **(iii)** for every $1 \leqslant i \leqslant n$, every proper left-factor of $u_i u_{i+1}$ is reduced.

Let $\underline{u} = u_0 | \cdots | u_n$ be an n-cell of $\mathrm{Std}(\mathrm{Red}_m(X))$ that is not critical, and let i be the minimal element of $\{0, \ldots, n\}$ such that $u_0 | \cdots | u_i$ is not critical. We say that $\underline{u}$ is

- **(i)** i-*subcritical* if $i \geqslant 1$ and $u_{i-1}u_i$ is reduced,
- **(ii)** i-*supercritical* if $i = 0$ (and, thus, $\|u_0\| > 1$), or $i \geqslant 1$ and $u_{i-1}u_i$ has a nonreduced proper left-factor.

If $\underline{u}$ is i-supercritical, then, u_{i-1} being reduced, there exists a proper factorisation $u_i = v_i w_i$ such that $u_{i-1}v_i$ it the shortest nonreduced left-factor of $u_{i-1}u_i$, and we put $\delta(\underline{u}) = (v_i, w_i)$.

7.6.4. Theorem (Squier's polygraphic resolution). *Let A be an augmented algebra and X be a reduced convergent left-monomial presentation of A. Define $\mathrm{Sq}(X)$ to be the graded subset of $\mathrm{Std}(\mathrm{Red}_m(X))$ consisting of its critical cells, and $\varphi : \mathrm{Sq}(X) \to \mathrm{Sq}(X)$ as the graded partial map of degree 1 given by*

$$\varphi(u_0|\cdots|u_n) = u_0|\cdots|u_{i-1}|v_i|w_i|u_{i+1}|\cdots|u_n$$

for every i-supercritical n-cell $u_0|\cdots|u_n$ with $\delta(u_0|\cdots|u_n) = (v_i, w_i)$. Then φ is a collapsing scheme of $\mathrm{Std}(\mathrm{Red}_m(X))$ onto $\mathrm{Sq}(X)$. As a consequence, $\mathrm{Sq}(X)$, equipped with the structure of ∞-polygraph induced by φ, is a polygraphic resolution of A.

Proof. First, φ induces a bijection between the i-supercritical n-cells and the $(i+1)$-subcritical $(n+1)$-cells of $\mathrm{Std}(\mathrm{Red}_m(X))$, with inverse given by $\psi(u_0|\cdots|u_{n+1}) = u_0|\cdots|u_iu_{i+1}|\cdots|u_n$. Second, we have $\mathrm{Std}(\mathrm{Red}_m(X)) = \mathrm{im}(\varphi) \sqcup \mathrm{Sq}(X) \sqcup \mathrm{dom}(\varphi)$ because $\mathrm{Sq}(X)$, $\mathrm{dom}(\varphi)$ and $\mathrm{im}(\varphi)$ respectively consist of the critical, supercritical and subcritical cells of $\mathrm{Std}(\mathrm{Red}_m(X))$. Third, let $\underline{u} = u_0|\cdots|u_n$ be an i-subcritical n-cell of $\mathrm{Std}(\mathrm{Red}_m(X))$, so that $\underline{u} = \varphi(d^+_{i+1}(\underline{u}))$. The $(n-1)$-cells that appear in $\partial(\underline{u})$ are $u_0|\cdots|u_{n-1}$, $u_1|\cdots|u_n$ and each $d^+_j(\underline{u})$, for $1 \leqslant j \leqslant n$, see [95, Corollary 6.1.3]. Since $d^+_{i+1}(\underline{u})$ is distinct from each other $(n-1)$-cell of $\mathrm{Std}(\mathrm{Red}_m(X))$ that appears in $\partial(\underline{u})$, we deduce that $\partial(\underline{u})$ has the required form.

Finally, we check that the relation $\trianglerighteq_\varphi$ induced by φ is wellfounded, by proving that it is included into the wellfounded order $\trianglerighteq$ defined by $u_0|\cdots|u_n \triangleright v_0|\cdots|v_n$ if either

- **(i)** $v_0\cdots v_n$ is a proper submonomial of $u_0\cdots u_n$, or
- **(ii)** $u_0\cdots u_n \succ_X a$, with a an n-cell of $\mathbb{K}\langle\mathrm{Std}(\mathrm{Red}_m(X))\rangle$ such that $v_0|\cdots|v_n$ belongs to $\mathrm{Supp}(a)$, or
- **(iii)** $u_0\cdots u_n = v_0\cdots v_n$, and there exists i in $\{0,\ldots,n\}$ such that $u_0 = v_0, \ldots, u_{i-1} = v_{i-1}$, and $l(u_i) > l(v_i)$.

Fix an i-subcritical n-cell $\underline{u} = u_0|\cdots|u_n$ of $\mathrm{Std}(\mathrm{Red}_m(X))$. Let us prove that $d^+_{i+1}(\underline{u}) \trianglerighteq y$ holds for every $(n-1)$-cell y of $\mathrm{Std}(\mathrm{Red}_m(X))$ that appears in the support of $u_0|\cdots|u_{n-1}$, $u_1|\cdots|u_n$ and of each $d^+_j(\underline{u})$, for $1 \leqslant j \leqslant n$ and $j \neq i+1$. In the first two cases, **(i)** applies. Otherwise, $y = u_0|\cdots|u_{j-1}|v|u_{j+2}|\cdots|u_n$ for $v \in \mathrm{Supp}(\widehat{u_iu_{i+1}})$. If u_ju_{j+1} is not reduced, then $u_0\cdots u_{j-1}vu_{j+2}\cdots u_n$ is not a proper submonomial of $u_0\cdots u_n$ because X terminates, so **(ii)** applies. If u_ju_{j+1} is reduced, then **(i)** and **(ii)** fail, but **(iii)** applies, because $u_0|\cdots|u_{i-1}$ critical implies that u_ku_{k+1} is nonreduced for each $k < i$, so that $j > i+1$ holds. □

We obtain the analogue of Theorem 4.4.3 for associative algebras, referring to [95, Theorem 7.1.3] for a complete proof:

7.6.5. Theorem. *Let X be an ∞-polygraph that presents an algebra A. Then the sequence*

$$A \xleftarrow{\mu} A\otimes A^o \xleftarrow{\delta_0} A\otimes\mathbb{K}X_0\otimes A \xleftarrow{\delta_1} \cdots \xleftarrow{\delta_k} A\otimes\mathbb{K}X_k\otimes A \xleftarrow{\delta_{k+1}} \cdots \qquad (7.8)$$

with $\mu(a\otimes b) = ab$, $\delta_0[x] = 1\otimes x - x\otimes 1$, and, for $k \geqslant 0$, $\delta_k[\alpha] = [s_{k-1}(\alpha)] - [t_{k-1}(\alpha)]$, is a chain complex in $\mathcal{B}imod(A)$. Moreover, if X is a polygraphic resolution of A, then (7.8) is a resolution of A by free A-bimodules.

7.6.6. Koszul algebras. All the notions studied so far can be generalised to the case of graded algebras by replacing the categories $\mathcal{V}\!ect$ and $\mathcal{A}\!lg$ by their graded versions $\mathcal{V}\!ect_{\mathrm{gr}}$ and $\mathcal{A}\!lg_{\mathrm{gr}}$. Given a fixed map $\omega : \mathbb{N}_+ \to \mathbb{N}_+$, we say that a graded ∞-polygraph X is ω-*concentrated* if each graded set X_n, for $n \geqslant 0$, is concentrated in degree $\omega(n+1)$. Fix $N \geqslant 2$, and define $\ell_N : \mathbb{N}_+ \to \mathbb{N}_+$ by $\ell_N(2p) = Np$ and $\ell_N(2p+1) = Np+1$. An N-homogeneous algebra A is said to be *Koszul* if there exists a resolution $M_* \twoheadrightarrow A$ of A by projective graded A-bimodules such that each M_n is generated by $M_n^{(\ell_N(n))}$.

From Theorem 7.6.5, we deduce the following conditions for an algebra to be Koszul or not:

7.6.7. Proposition. *Fix $N \geqslant 2$, and let A be an N-homogeneous algebra.*

(i) *If A admits an ℓ_N-concentrated polygraphic resolution, then A is Koszul.*

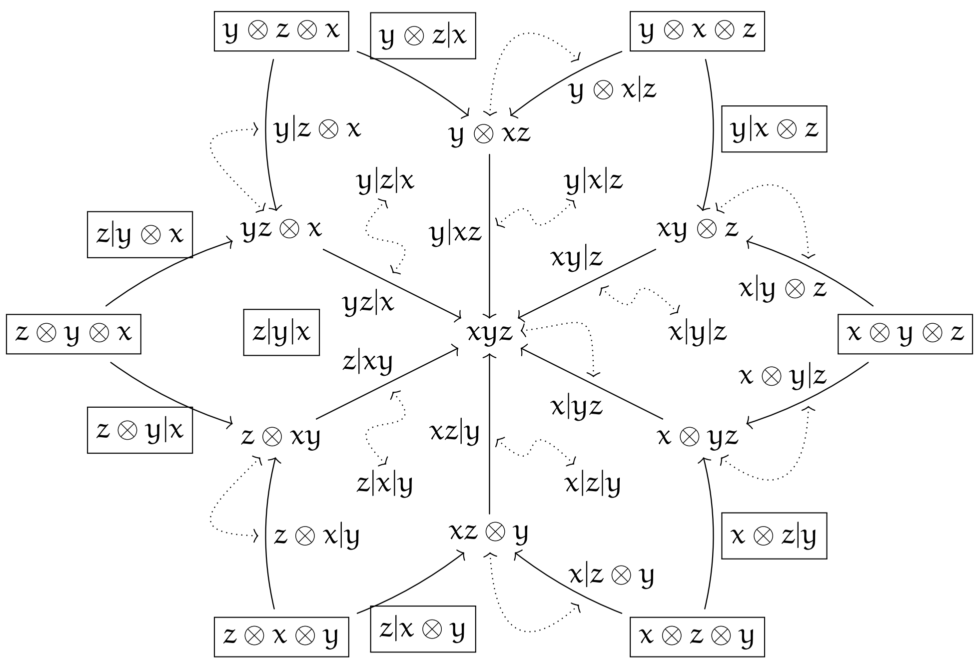
(ii) *If A admits a quadratic convergent presentation, then A is Koszul.*

(iii) *If A admits a polygraphic presentation X such that, for some $n \geqslant 1$, X_n is ℓ_N-concentrated but $X_n^{(i)}$ has strictly more elements than $X_{n+1}^{(i)}$ for some $i > \ell_N(n+1)$, then A is not Koszul.*

7.6.8. Example. Consider the symmetric algebra on a totally ordered set X, presented by

$$\mathrm{Sym}(x, y, z) = \left(X \,\middle|\, (y|x : yx \to xy)_{x<y\in X}\right).$$

The 1-polygraph $\mathrm{Sym}(X)$ is left-monomial and convergent, so that Theorem 7.6.4 applies to it: we obtain a polygraphic resolution $\mathrm{Sq}(\mathrm{Sym}(X))$ with one n-cell $x_n|\cdots|x_0$ for each sequence $x_n > \cdots > x_0$ in X. The boundary of these cells are obtained by collapsing the standard polygraphic resolution, through the collapsing scheme φ of Theorem 7.6.4. In the case of a 3-cell $z|y|x$, the cells required to understand the computation are drawn on the following figure:



The identifications induced by φ are given by the dotted arrows, and the cells that belong to $\mathrm{Sq}(\mathrm{Sym}(x, y, z))$ are boxed. Let us explain how these boxed cells and their boundary are obtained.

In the figure, all the cells belong to the standard polygraphic resolution. The six 2-cells of the figure correspond to all the permutations of the generators x, y and z. But, among these 2-cells, only one is critical, $z|y|x$, all the other ones being subcritical: for example, $y|z|x$ is subcritical because $y|z$ is reduced. Now, in the standard polygraphic, the source of $z|y|x$ is

$$z|y \otimes x \quad \star_0 \quad yz|x.$$

The 1-cell $z|y$ is critical, but $yz|x$ is supercritical, because yz has length 2. So, to obtain the source of $z|y|x$ in $\mathrm{Sq}(\mathrm{Sym}(X))$, we have to apply to it the functor induced by the collapsing scheme φ, replacing $yz|x$ by the rest of the boundary of $y|z|x$, which is

$$(y|z)^- \otimes x \quad \star_0 \quad y \otimes z|x \quad \star_0 \quad y|xz.$$

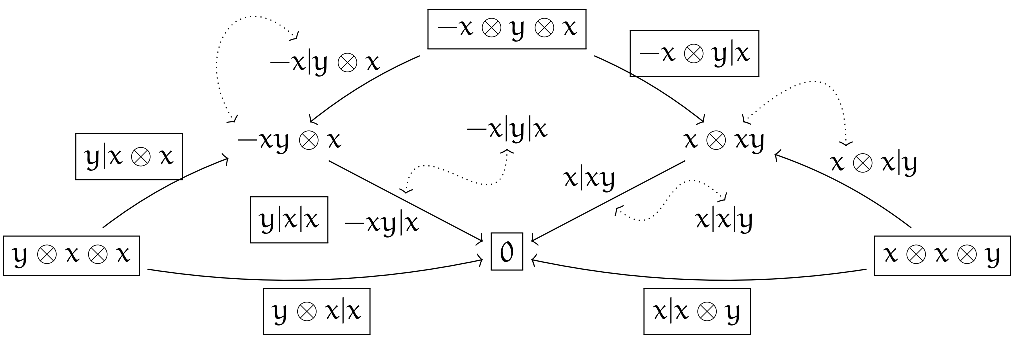
Among the 1-cells involved in this last composition, only $z|x$ is critical: $y|z$ is subcritical, and thus will be collapsed to an identity, while $y|xz$ is supercritical, and thus must also be iteratively replaced. The process continues until there remains only critical cells, which is guaranteed by the fact that φ is a collapsing scheme.

Finally, applying Proposition 7.6.7 to the polygraphic resolution so obtained, we recover the Koszulness of the symmetric algebra. In fact, a detailed study shows that the complex given by Theorem 7.6.5 applied to $\mathrm{Sq}(\mathrm{Sym}(x, y, z))$ is, up to the sign of the differential, the usual Koszul complex of the symmetric algebra, see [95, Examples 6.3.2 and 7.1.4].

7.6.9. Example. Consider the exterior algebra on a totally ordered set X, presented by

$$\mathrm{Ext}(X) = \Big(X \,\Big|\, (y|x : yx \to xy)_{x<y\in X},\ (x|x : xx \to 0)_{x\in X} \Big).$$

This 1-polygraph is left-monomial and convergent, and Theorem 7.6.4 gives a polygraphic resolution of the exterior algebra with one n-cell $x_n|\cdots|x_0$ for each sequence $x_n \geqslant \cdots \geqslant x_0$ in X. For example, the boundary of a 3-cell $y|y|x$, for $x < y$, is given by the following collapsing:



As a consequence of Proposition 7.6.7, we recover the Koszulness of the exterior algebra.

Chapter 8
Perspectives

8.1. Higher categories and polygraphs

The first research direction concerns the theoretical setting of Squier theory: higher categories, polygraphs, and rewriting theory. Here, the goal is to look for more general definitions of higher categories that encompass the maximum of variants (shapes of cells, strictness, with inverses or not, with an underlying algebraic structure or not, the latter being strict or not), with a corresponding notion of polygraphs and theory of rewriting.

8.1.1. Higher categories in categories of monoids. With Marcelo Fiore, we currently explore a new point of view on higher categories. Our starting observation is that spans internal to monoids in a monoidal category are equipped with two products, span composition, and another one induced by the monoid product, and these two products satisfy an interchange law. Thus, these spans form a *duoidal* category, as defined in [3] under the name of 2-monoidal category: a category equipped with two compatible monoidal structures. In particular, 2-categories are *duoids* in a duoidal category of 2-globular objects, iterated spans form an n-*oidal* category, and internal n-categories are n-*oids* in an n-oidal category of n-globular objects. This leads to a new definition of internal n-categories, itself inducing a definition of polygraph for such higher categories. This point of view applies to monoids, monoidal categories, and associative algebras, but also to operads, Lawvere theories, and higher-order theories like the λ-calculus, when formulated in the spirit of [73]. Thus, we can study polygraphic resolutions of symmetric operads and algebras over them, as started in collaboration with Emily Burgunder, Bérénice Delcroix-Oger and Joan Millès, or of objects like the λ-calculus, which is a common project with Marcelo Fiore and Damiano Mazza to bring a new light on standardisation theory [82]. Another requirement to develop a general Squier theory is to find a model structure on the resulting higher categories; a preliminary exploration with Dimitri Ara has shown that the canonical model structure on higher categories (internal to sets) should transpose to higher categories internal to categories of algebras over a projective sketch, in the same way as it was transposed to higher groupoids in [9].

8.1.2. Weakening higher categories. All the higher categories considered in this document are strict: the relations satisfied by the composition hold strictly, and so do the ones of the ambient algebraic structure (monoid or algebra). The motivation for considering weakened versions is that they should produce cofibrant approximations that would be excellent candidates for formalisations of algebraic objects, for example as higher inductive types in type theory. Weakening all the relations at the same time is too ambitious, but it seems possible in a progressive manner. First, relaxing the ambient algebraic structure relies both on a better combinatorial understanding of the globular Gray tensor product of higher categories (work in progress with François Métayer), and on a computation of all the coherences of the algebraic structure, such as all associahedra for associative structures. Next, before totally relaxing the axioms of higher categories, one should consider semistrict versions, as Simon Forest has started to explore in his PhD thesis [74]. A complete weakening may require a totally different point of view, such as the one provided by homotopy type theory: in his PhD thesis, Antoine Allioux explores ways to formalise weak monoids in homotopy type theory, a first step in that direction.

8.1.3. Controlling the shapes of cells. Following observations by Maxime Lucas, I want to explore a different description of globular, simplicial and cubical objets in a category that is based on the symmetry groups of the cells of various dimensions, rather than on the explicit data of their face maps. In such a formulation, the globular n-cells would correspond to the Coxeter type A_1^n, the simplicial ones to A_n, and the cubical ones to B_n. This should permit to control the shapes of cells by a parameter (the Coxeter types in all dimensions) and to adopt a generic, shape-independent point of view on higher categories. This would lead to new notions of higher categories, with simplicial cells, for example, and an equivalence such as the one between globular and cubical higher categories [2] would follow from a connexion between the corresponding Coxeter types.

8.1.4. A more abstract algebraic rewriting theory. In the case of associative algebras, the isomorphism between ∞-categories and ∞-groupoids internal to $\mathcal{Alg}$ shows that major obstructions may arise when trying to develop a rewriting theory for a given algebraic structure. For associative algebras, we could circumvent this issue by introducing an adapted notion of rewriting step, but this does not seem to be the same for groups, or for algebras over a symmetric operad. Another point of view, still inspired by the case of algebras, would be to place ourselves directly in an ∞-groupoid $X^\top$ internal to a concrete category $\mathcal{C}$, and trying to identity a set-theoretic subcategory of the 1-groupoid underlying $X^\top$ that will play the role of the positive 1-cells: for example, the positiveness could depend on a global order that, contrarily to the case of Gröbner bases or Poincaré-Birkhoff-Witt bases, needs not be total.

8.2. Garside methods in rewriting and in Squier theory

The constructions presented in this document rely on convergent presentations. However, experience shows that finite convergent presentations are difficult to find for many interesting examples, and concrete computations of polygraphic resolutions are generally hard. Here, the general objective is to integrate methods from Garside theory to help on these two points.

8.2.1. Polygraphic resolutions from Garside families. In an almost completed work with Matthieu Picantin, we use Garside families to compute more explicit polygraphic resolutions of monoids than the ones given by a generic convergent presentation. However, we use a different notion of polygraph: differential graded polygraphs, which generate quasifree differential graded algebras (dg-algebras for short) [117, 142]. This allows us to focus on the computation of the cells of the resolution, rather than on the bureaucracy involved by their globular description. As a result, given a monoid M with a Garside family X, we obtain a dg-polygraphic resolution of the algebra $\mathbb{K}M$, with a simple description of the generating cells and of their boundary. Moreover, if M is atomic, we can contract the dg-polygraphic resolution into a smaller one. After abelianisation of these two resolutions, we obtain generalisations of the second and third resolutions of [64]. The next step will be to translate these constructions into polygraphic resolutions in $\infty\mathcal{G}pd(\mathcal{M}on)$, which will generalise the results of [78] in all dimensions, and to any monoid with a Garside family or any atomic monoid. This will require a better understanding of the shapes of n-cells generated by quadratic normalisations of class $(4, 3)$, that we call n-silexes, starting with an heptagon in dimension 2, and a silex-looking polytope in dimension 3.

8.2.2. The KGB completion procedure. As proved by Kapur and Narendran in [122], the existence of a finite convergent presentation of a monoid M depends on the generators: applying Knuth-Bendix completion procedure to Artin's presentation of B_3^+ will never terminate, because it only adds relations and never changes the generators. As far as I know, only three proposals have been made to enhance completion procedures with such an ability. First, for term rewriting systems, Knuth and Bendix in [126] examine the variables that appear in the two normal forms of a nonconfluent critical branching; if they are not the same, they introduce a new generating symbol that only depends on the common variables. Second, Pedersen's morphocompletion was introduced in [174]: it tries to apply standard completion for a number of runs and, if confluence is not reached, backtracks and adds new generators to close the existing nonconfluent critical branchings. Third, with Philippe Malbos and Samuel Mimram, we observed in [101] that the generators of Garside's presentations for Artin monoids and of the column presentations of plactic and Chinese monoids generate all the quasicenters of all the parabolic submonoids, and we proposed to enhance completion by computing these. Still with Matthieu Picantin, we currently explore a fourth enhanced completion procedure, called KGB for "Knuth-Bendix with Garside inside". It originates in the result of [63] asserting that every finite Garside family gives rise to a finite convergent presentation. Moreover, provided some mild hypotheses are satisfied, Garside families can be defined as generating families that are closed under simple operations such as common multiple and complement. The idea of the KGB procedure is to interleave the computations of a Garside family and of a convergent presentations, and practical experiments already validate the procedure on low-rank graded examples such as Artin monoids.

8.2.3. Generalising Garside theory. Garside theory is a theory of normal forms in monoids, and also generalises in a straightforward way to categories [60]. But it seems that Garside theory could also be developed, in an abstract setting, for monoids in a monoidal category with some extra properties, such as being concrete. Doing so would give a totally new way to compute normal forms, and thus resolutions, for several monoidal structures such as pros, operads or Lawvere theories. The main question is not whether it is possible, but whether there is evidence that such objects could admit nontrivial Garside families. The first example that I wish to investigate, with Julien Ross, comes from the search for normal forms of quantum circuits with specific sets of gates: Clifford in [189] and CNOT+T in [4]. In both cases, the computation of normal forms rely on the adjunction of extra redundant generators, sharing common properties with Garside families, such as forming a closed family with respect to complement and common multiple.

8.2.4. Convergence revisited. As already mentioned, reaching convergence is hard for many interesting examples outside the world of monoids. The main issue is termination, which is a strong requirement, and serves in Squier theory for inductions on the length of the sequences of rewriting steps: weaker requirements could still allow this, like, for example, asking noetherianity of left or right-divisibility in the category $\overline{\mathsf{U}}(X^*)$ obtained by identifying all parallel 1-cells of X^*. At the same time, concrete examples usually satisfy a stronger property than confluence: two 1-cells with the same source have a minimal way, or at least a locally minimal way, to be completed into a 1-sphere. Said differently, in the category $\overline{\mathsf{U}}(X^*)$, two 1-cells with the same source have at least a minimal common multiple. This stronger hypothesis would remove the need to consider negative cells when building an acyclic extension from critical branchings. So, using concepts central to Garside theory, like divisibility or minimal common multiples, to relax termination while strengthening confluence, seems to better fit concrete examples, and should produce an alternative notion of convergence that is more adapted to compute polygraphic resolutions.

8.3. Polygraphic resolutions of Artin monoids and groups

In [78], we have shown how to obtain the first dimensions of polygraphic resolutions of Artin monoids. Moreover, further exploration seems to indicate that the corresponding complete polygraphic resolutions have the same cells as well-known CW complexes associated to Artin monoids and related to the $K(\pi, 1)$-conjecture, see e.g. [47]: the standard polygraphic resolution of $B^+(W)$ and its geometric realisation; Garside's polygraphic resolution and Deligne's complex; the unfolding [137] of Artin polygraphic resolution and the universal covering of Salvetti's complex. Hence, we think that the $K(\pi, 1)$-conjecture is equivalent to the fact that any of the standard, Garside's or Artin's polygraphic resolution is also one for the Artin *group* $B(W)$.

8.3.1. Concrete computations of polygraphic resolutions. As already noted, the construction of the polygraphic resolution in [99] is not explicit enough to apply the contraction method of [78] in higher dimensions. This is the main motivation behind my recent interest into Garside theory. Also, the standard resolution could be defined explicitly in [95] in the case of associative algebras, thanks to the underlying linear structure, but formulas still have to be given in the set-theoretic case. Our planned exploration of the Gray tensor product of globular higher categories, with François Métayer, will provide a way to compute this standard polygraphic resolution. Then, a set-theoretic collapsing scheme should yield Garside's and Artin's polygraphic resolutions.

8.3.2. Geometric realisation of polygraphs. It is well-known that strict higher categories do not model all homotopy types. Therefore, it is not possible to define a geometric realisation functor so that a strict higher category has exactly the same homotopical properties as its realisation. However, in a recent work on Simpson's conjecture [106], Simon Henry has introduced a notion of regular polygraphs, that are sufficiently well-formed to admit a geometric realisation. Note that these regular polygraphs are almost the same as the ones defined independently and at the same time by Amar Hadzihasanovic [103]. With Simon Henry, we conjecture that the three polygraphs that interest us in the case of Artin monoids are regular in his sense, and thus admit geometric realisations with the same homotopical properties: we plan to investigate this further, and to relate the resulting spaces to the corresponding known CW complexes.

8.3.3. Polygraphic resolutions of groups. The concept of polygraphic resolution can be adapted to groups in a straightforward way: given a group G, a polygraphic resolution of G is a cofibrant approximation of G in the category of ∞-groupoids. Now, if M is a monoid and X is a polygraphic resolution of M, one can see X as an $(\infty, 0)$-polygraph and wonder if it is also a polygraphic resolution of the enveloping group $G(M)$ of M: this has no reason to be true in general, because the introduction of inverses may create spheres in the free ∞-groupoid over X that were not present in the free $(\infty, 1)$-category over X. However, if all the previous steps are valid, then the $K(\pi, 1)$-conjecture says precisely that this is the case for $M = B^+(W)$. We know that every Artin monoid injects in the corresponding Artin group [173], but the axiomatics of model categories do not permit to exploit that fact directly: an investigation of other respective properties of Artin monoids and groups, and of derived objects such as Artin dual monoids, is necessary to hope for further progress here.

8.4. Algebraic invariants of computation

Starting in 2008, my activity has shifted to the use of rewriting methods in effective algebra. However, I was interested before in the use of algebraic methods in theoretical computer science, and some research directions that I still would like to explore some day belong to that family.

8.4.1. Beyond finite convergent presentations. Squier has proved in 1987 that, if the existence of a finite convergent presentation for a monoid implies the decidability of its word problem, the converse is not true. Then, in 1998, Otto, Katsura and Kobayashi showed an equivalence [171]: a finitely generated monoid has a decidable word problem if, and only if, it admits a *left-recursive* convergent presentation; here, left-recursive means that the set of sources of the relations form a recursive language. Moreover, they showed that replacing left-recursive by a more strict condition, such as left-regular, breaks the equivalence again. This suggests several research directions: homological and homotopical invariants adapted to left-C convergent presentations, where C is a class of formal languages; a similar result for Lawvere theories, term rewriting systems and formal languages on trees; the exploration of left-regular convergent presentations, a good compromise between finite and left-recursive ones containing all known counterexamples of monoids with a decidable word problem but no finite convergent presentation: Kapur-Narendran, Squier, Lafont-Prouté.

8.4.2. Cohomology and complexity. To conclude with a science-fiction research direction, the termination-by-derivation method of [88, 89] has been later adapted into a tool for complexity analysis [30, 31]. This led to a characterisation of the complexity class FP in polygraphic terms: a function f is in FP if, and only if, there exists a polygraphic program X computing f and admitting a derivation d into a certain module P over X^*. But, for a given algebraic object A, the first group $H^1(A, M)$ of the usual associated cohomology theory classifies the derivations of A into M. This suggests an interesting possible application of cohomological methods in complexity theory.

Bibliography

[1] Sergey I. Adyan, *Fragments of the word Δ in the braid group*, Mat. Zametki 36.1 (1984), pp. 25–34 (cit. on p. 85).

[2] Fahd Ali Al-Agl, Ronald Brown, and Richard Steiner, *Multiple categories: the equivalence of a globular and a cubical approach*, Adv. Math. 170.1 (2002), pp. 71–118 (cit. on pp. 13, 122).

[3] Marcelo Aguiar and Swapneel Mahajan, *Monoidal functors, species and Hopf algebras*, vol. 29, CRM Monograph Series, With forewords by Kenneth Brown and Stephen Chase and André Joyal, American Mathematical Society, Providence, RI, 2010, pp. lii+784 (cit. on p. 121).

[4] Matthew Amy, Jianxin Chen, and Neil J. Ross, *A finite presentation of CNOT-dihedral operators*, Proceedings 14th International Conference on Quantum Physics and Logic, vol. 266, Electron. Proc. Theor. Comput. Sci. (EPTCS), EPTCS, [place of publication not identified], 2018, pp. 84–97 (cit. on p. 123).

[5] David J. Anick, *On monomial algebras of finite global dimension*, Trans. Amer. Math. Soc. 291.1 (1985), pp. 291–310 (cit. on p. 102).

[6] David J. Anick, *On the Homology of Associative Algebras*, Trans. Amer. Math. Soc. 296.2 (1986), pp. 641–659 (cit. on pp. 49, 61, 102).

[7] David J. Anick and Edward L. Green, *On the homology of quotients of path algebras*, Comm. Algebra 15.1-2 (1987), pp. 309–341 (cit. on p. 102).

[8] Dimitri Ara and Georges Maltsiniotis, *Le type d'homotopie de la ∞-catégorie associée à un complexe simplicial*, arXiv:1503.02720, 2015 (cit. on pp. 50, 53).

[9] Dimitri Ara and François Métayer, *The Brown-Golasiński model structure on strict ∞-groupoids revisited*, Homology Homotopy Appl. 13.1 (2011), pp. 121–142 (cit. on pp. 8, 50, 52, 121).

[10] Franz Baader and Tobias Nipkow, *Term rewriting and all that*, Cambridge University Press, 1998 (cit. on pp. 7, 109).

[11] Jörgen Backelin, *A distributiveness property of augmented algebras, and some related homological results*, PhD thesis, Stockholm University, 1983 (cit. on p. 102).

[12] Jörgen Backelin, Svetlana Cojocaru, and Victor Ufnarovski, *The computer algebra package Bergman: current state*, Commutative algebra, singularities and computer algebra (Sinaia, 2002), vol. 115, NATO Sci. Ser. II Math. Phys. Chem. Kluwer Acad. Publ., Dordrecht, 2003, pp. 75–100 (cit. on p. 102).

[13] Jörgen Backelin and Ralf Fröberg, *Koszul algebras, Veronese subrings and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. 30.2 (1985), pp. 85–97 (cit. on p. 102).

[14] John Baez, *Higher-dimensional algebra. II. 2-Hilbert spaces*, Adv. Math. 127.2 (1997), pp. 125–189 (cit. on p. 68).

[15] John C. Baez and Alissa S. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, Theory Appl. Categ. 12 (2004), pp. 492–538 (cit. on pp. 68, 103).

[16] John C. Baez and James Dolan, *Higher-dimensional algebra. III. n-categories and the algebra of opetopes*, Adv. Math. 135.2 (1998), pp. 145–206 (cit. on p. 13).

[17] Michael A. Batanin, *Computads for finitary monads on globular sets*, Higher category theory (Evanston, IL, 1997), vol. 230, Contemp. Math. Amer. Math. Soc., Providence, RI, 1998, pp. 37–57 (cit. on pp. 14, 103).

[18] Hans Joachim Baues, *Combinatorial homotopy and 4-dimensional complexes*, vol. 2, De Gruyter Expositions in Mathematics, Berlin: De Gruyter, 1991, pp. xxviii+380 (cit. on pp. 13, 32).

[19] Hans Joachim Baues and Winfried Dreckmann, *The cohomology of homotopy categories and the general linear group*, K-Theory 3.4 (1989), pp. 307–338 (cit. on p. 32).

[20] Hans Joachim Baues and Günther Wirsching, *Cohomology of small categories*, J. Pure Appl. Algebra 38.2-3 (1985), pp. 187–211 (cit. on p. 14).

[21] Hans-Joachim Baues and Mamuka Jibladze, *Classification of abelian track categories*, K-Theory 25.3 (2002), pp. 299–311 (cit. on p. 32).

[22] Hans-Joachim Baues and Elias Gabriel Minian, *Track extensions of categories and cohomology*, K-Theory 23.1 (2001), pp. 1–13 (cit. on p. 32).

[23] Jean Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77 (cit. on p. 13).

[24] Roland Berger, *Confluence and Koszulity*, J. Algebra 201.1 (1998), pp. 243–283 (cit. on p. 102).

[25] Roland Berger, *Koszulity for nonquadratic algebras*, J. Algebra 239.2 (2001), pp. 705–734 (cit. on p. 102).

[26] Roland Berger and Nicolas Marconnet, *Koszul and Gorenstein properties for homogeneous algebras*, Algebr. Represent. Theory 9.1 (2006), pp. 67–97 (cit. on p. 102).

[27] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. 29.2 (1978), pp. 178–218 (cit. on p. 101).

[28] Leonid A. Bokut, *Imbeddings into simple associative algebras*, Algebra i Logika 15.2 (1976), pp. 117–142, 245 (cit. on p. 101).

[29] Leonid A. Bokut, Yuqun Chen, Weiping Chen, and Jing Li, *New approaches to plactic monoid via Gröbner-Shirshov bases*, J. Algebra 423 (2015), pp. 301–317 (cit. on pp. 85, 95).

[30] Guillaume Bonfante and Yves Guiraud, *Intensional properties of polygraphs*, Electronic Notes in Theoretical Computer Science 203.1 (2008), pp. 65–77 (cit. on pp. 8, 14, 125).

[31] Guillaume Bonfante and Yves Guiraud, *Polygraphic programs and polynomial-time functions*, Logical Methods in Computer Science 5.2:14 (2009), pp. 1–37 (cit. on pp. 8, 14, 125).

[32] Ronald Vernon Book and Friedrich Otto, *String-rewriting systems*, Texts and Monographs in Computer Science, Springer-Verlag, 1993 (cit. on p. 7).

[33] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V : Groupes engendrés par des réflexions. Chapitre VI : systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Paris: Hermann, 1968, 288 pp. (Cit. on p. 82).

[34] Dominique Bourn, *Another denormalization theorem for abelian chain complexes*, J. Pure Appl. Algebra 66.3 (1990), pp. 229–249 (cit. on p. 103).

[35] Murray Bremner and Vladimir Dotsenko, *Algebraic Operads: An Algorithmic Companion*, Taylor & Francis Group, 2016 (cit. on p. 102).

[36] Egbert Brieskorn and Kyoji Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. 17 (1972), pp. 245–271 (cit. on pp. 69, 78, 79).

[37] Kenneth S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), vol. 23, Math. Sci. Res. Inst. Publ. New York: Springer, 1992, pp. 137–163 (cit. on pp. 49, 69, 105).

[38] Ronald Brown and Philip J. Higgins, *The equivalence of ∞-groupoids and crossed complexes*, Cahiers Topologie Géom. Différentielle 22.4 (1981), pp. 371–386 (cit. on p. 13).

[39] Ronald Brown and Philip J. Higgins, *The equivalence of ω-groupoids and cubical T-complexes*, Cahiers Topologie Géom. Différentielle 22.4 (1981), pp. 349–370 (cit. on p. 13).

[40] Ronald Brown and Johannes Huebschmann, *Identities among relations*, Low-dimensional topology (Bangor, 1979), vol. 48, London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 1982, pp. 153–202 (cit. on pp. 32, 49).

[41] Bruno Buchberger, *History and basic features of the critical-pair/completion procedure*, J. Symbolic Comput. 3.1-2 (1987), Rewriting techniques and applications (Dijon, 1985), pp. 3–38 (cit. on p. 101).

[42] Bruno Buchberger, *An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal*, J. Symbolic Comput. 41.3-4 (2006), Translated from the 1965 German original by Michael P. Abramson, pp. 475–511 (cit. on pp. 7, 101, 110).

[43] Albert Burroni, *Higher-dimensional word problems with applications to equational logic*, Theoret. Comput. Sci. 115.1 (July 1993), pp. 43–62 (cit. on pp. 7, 8, 13, 14, 31).

[44] Alan J. Cain, Robert D. Gray, and António Malheiro, *Finite Gröbner-Shirshov bases for plactic algebras and biautomatic structures for plactic monoids*, J. Algebra 423 (2015), pp. 37–53 (cit. on pp. 85, 95).

[45] Alan J. Cain, Robert D. Gray, and António Malheiro, *Rewriting systems and biautomatic structures for Chinese, hypoplactic, and Sylvester monoids*, Internat. J. Algebra Comput. 25.1-2 (2015), pp. 51–80 (cit. on p. 97).

[46] Julien Cassaigne, Marc Espie, Daniel Krob, Jean-Christophe Novelli, and Florent Hivert, *The Chinese monoid*, Internat. J. Algebra Comput. 11.3 (2001), pp. 301–334 (cit. on p. 97).

[47] Ruth Charney and Michael W. Davis, *Finite* $\mathsf{K}(\pi, 1)$*s for Artin groups*, Prospects in topology (Princeton, NJ, 1994), vol. 138, Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1995, pp. 110–124 (cit. on p. 124).

[48] Cyrille Chenavier, *Confluence algebras and acyclicity of the Koszul complex*, Algebr. Represent. Theory 19.3 (2016), pp. 679–711 (cit. on p. 102).

[49] Cyrille Chenavier, *The lattice of reduction operators: applications to noncommutative Gröbner bases and homological algebra*, Theses, Université paris Diderot, 2016 (cit. on p. 102).

[50] Cyrille Chenavier, *A Lattice Formulation of the F4 Completion Procedure*, preprint, 2018 (cit. on p. 102).

[51] Cyrille Chenavier, *Reduction operators and completion of rewriting systems*, J. Symbolic Comput. 84 (2018), pp. 57–83 (cit. on p. 102).

[52] Cyrille Chenavier, *Syzygies among reduction operators*, J. Pure Appl. Algebra 223.2 (2019), pp. 721–737 (cit. on p. 102).

[53] Eugenia Cheng and Aaron Lauda, *Higher-dimensional categories: an illustrated guide book*, 2004 (cit. on p. 13).

[54] Alonzo Church, *The Calculi of Lambda-Conversion*, Annals of Mathematics Studies, no. 6, Princeton University Press, Princeton, N. J., 1941, pp. ii+77 (cit. on p. 7).

[55] Daniel E. Cohen, *A monoid which is right* FP_∞ *but not left* FP_1, Bull. London Math. Soc. 24.4 (1992), pp. 340–342 (cit. on pp. 49, 61).

[56] Paul M. Cohn, *Universal algebra*, Harper & Row, Publishers, New York-London, 1965, pp. xv+333 (cit. on p. 101).

[57] Robert Cremanns and Friedrich Otto, *Finite derivation type implies the homological finiteness condition* FP_3, J. Symbolic Comput. 18.2 (1994), pp. 91–112 (cit. on pp. 10, 49, 61).

[58] Robert Cremanns and Friedrich Otto, *For groups the property of having finite derivation type is equivalent to the homological finiteness condition* FP_3, J. Symbolic Comput. 22.2 (1996), pp. 155–177 (cit. on pp. 32, 49).

[59] Patrick Dehornoy, *Groupes de Garside*, Ann. Sci. École Norm. Sup. (4) 35.2 (2002), pp. 267–306 (cit. on p. 70).

[60] Patrick Dehornoy, François Digne, Eddy Godelle, Daan Krammer, and Jean Michel, *Foundations of Garside theory*, vol. 22, EMS Tracts in Mathematics, Author name on title page: Daan Kramer, European Mathematical Society (EMS), Zürich, 2015, pp. xviii+691 (cit. on pp. 70, 85, 87, 97–99, 123).

[61] Patrick Dehornoy, Matthew Dyer, and Christophe Hohlweg, *Garside families in Artin-Tits monoids and low elements in Coxeter groups*, C. R. Math. Acad. Sci. Paris 353.5 (2015), pp. 403–408 (cit. on pp. 99, 100).

[62] Patrick Dehornoy and Volker Gebhardt, *Algorithms for Garside calculus*, J. Symbolic Comput. 63 (2014), pp. 68–116 (cit. on p. 85).

[63] Patrick Dehornoy and Yves Guiraud, *Quadratic normalisation in monoids*, Internat. J. Algebra Comput. 26.5 (2016), pp. 935–972 (cit. on pp. 11, 85, 86, 88, 89, 91, 92, 95, 96, 99, 123).

[64] Patrick Dehornoy and Yves Lafont, *Homology of Gaussian groups*, Ann. Inst. Fourier (Grenoble) 53.2 (2003), pp. 489–540 (cit. on p. 123).

[65] Patrick Dehornoy and Luis Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. London Math. Soc. (3) 79.3 (1999), pp. 569–604 (cit. on p. 70).

[66] Pierre Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972), pp. 273–302 (cit. on p. 69).

[67] Pierre Deligne, *Action du groupe des tresses sur une catégorie*, Invent. Math. 128.1 (1997), pp. 159–175 (cit. on pp. 67–70, 80, 83).

[68] Vladimir Dotsenko and Anton Khoroshkin, *Gröbner bases for operads*, Duke Math. J. 153.2 (2010), pp. 363–396 (cit. on p. 102).

[69] Josep Elgueta, *Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces*, Adv. Math. 213.1 (2008), pp. 53–92 (cit. on p. 68).

[70] Ben Elias and Geordie Williamson, *Soergel calculus*, arXiv:1309.0865, 2013 (cit. on p. 82).

[71] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992, pp. xii+330 (cit. on p. 85).

[72] Jean-Charles Faugère, *A new efficient algorithm for computing Gröbner bases* (F_4), J. Pure Appl. Algebra 139.1-3 (1999), Effective methods in algebraic geometry (Saint-Malo, 1998), pp. 61–88 (cit. on p. 102).

[73] Marcelo Fiore, Gordon Plotkin, and Daniele Turi, *Abstract syntax and variable binding (extended abstract)*, 14th Symposium on Logic in Computer Science (Trento, 1999), IEEE Computer Soc., Los Alamitos, CA, 1999, pp. 193–202 (cit. on p. 121).

[74] Simon Forest and Samuel Mimram, *Coherence of Gray Categories via Rewriting*, 3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018), ed. by Hélène Kirchner, vol. 108, Leibniz International Proceedings in Informatics (LIPIcs), Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018, 15:1–15:16 (cit. on p. 122).

[75] Ralph H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) 57 (1953), pp. 547–560 (cit. on p. 51).

[76] Nora Ganter and Mikhail Kapranov, *Representation and character theory in 2-categories*, Adv. Math. 217.5 (2008), pp. 2268–2300 (cit. on p. 68).

[77] Frank A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford Ser. (2) 20 (1969), pp. 235–254 (cit. on pp. 68, 69).

[78] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos, *Coherent presentations of Artin monoids*, Compos. Math. 151 (5 2015), pp. 957–998 (cit. on pp. 11, 67, 69–71, 75, 80, 81, 85, 87, 123, 124).

[79] Meinolf Geck, *PyCox: computing with (finite) Coxeter groups and Iwahori-Hecke algebras*, arXiv:1201.5566, 2012 (cit. on p. 82).

[80] Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, vol. 21, London Mathematical Society Monographs. New Series, New York: The Clarendon Press Oxford University Press, 2000, pp. xvi+446 (cit. on pp. 69, 79).

[81] Victor Ginzburg and Mikhail Kapranov, *Koszul duality for operads*, Duke Math. J. 76.1 (1994), pp. 203–272 (cit. on p. 31).

[82] Georges Gonthier, Jean-Jacques Lévy, and Paul-André Melliès, *An abstract standardisation theorem*, Proceedings of the Seventh Annual Symposium on Logic in Computer Science (LICS '92), Santa Cruz, California, USA, June 22-25, 1992, 1992, pp. 72–81 (cit. on p. 121).

[83] Robert Gordon, A. John Power, and Ross Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. 117.558 (1995), pp. vi+81 (cit. on p. 13).

[84] Hans Grauert, *Über die Deformation isolierter Singularitäten analytischer Mengen*, Invent. Math. 15 (1972), pp. 171–198 (cit. on p. 101).

[85] Edward L. Green, *Noncommutative Gröbner bases, and projective resolutions*, Computational methods for representations of groups and algebras (Essen, 1997), vol. 173, Progr. Math. Birkhäuser, Basel, 1999, pp. 29–60 (cit. on p. 102).

[86] Edward L. Green and Rosa Q. Huang, *Projective resolutions of straightening closed algebras generated by minors*, Adv. Math. 110.2 (1995), pp. 314–333 (cit. on p. 102).

[87] John R. J. Groves, *Rewriting systems and homology of groups*, Groups—Canberra 1989, vol. 1456, Lecture Notes in Math. Berlin: Springer, 1990, pp. 114–141 (cit. on p. 49).

[88] Yves Guiraud, *Présentations d'opérades et systèmes de réécriture*, PhD thesis, Université Montpellier 2, June 2004 (cit. on pp. 8, 11, 14, 15, 31, 125).

[89] Yves Guiraud, *Termination orders for 3-dimensional rewriting*, J. Pure and Appl. Algebra 207.2 (Oct. 2006), pp. 341–371 (cit. on pp. 8, 11, 14, 15, 31, 125).

[90] Yves Guiraud, *Termination orders for 3-polygraphs*, Comptes-Rendus de l'Académie des Sciences - Série I 342.4 (Feb. 2006), pp. 219–222 (cit. on pp. 8, 11, 14).

[91] Yves Guiraud, *The three dimensions of proofs*, Annals of Pure and Applied Logic 141.1-2 (Aug. 2006), pp. 266–295 (cit. on pp. 8, 14).

[92] Yves Guiraud, *Two polygraphic presentations of Petri nets*, Theoretical Computer Science 360.1-3 (Aug. 2006), pp. 124–146 (cit. on pp. 8, 14).

[93] Yves Guiraud, *Polygraphs for termination of left-linear term rewriting systems*, preprint, 2007 (cit. on p. 8).

[94] Yves Guiraud, *A library for the computation of coherent presentations of Artin monoids*, Python library, 2013 (cit. on p. 82).

[95] Yves Guiraud, Eric Hoffbeck, and Philippe Malbos, *Convergent presentations and polygraphic resolutions of associative algebras*, Math. Z. (2017), to appear, pp. 1–65 (cit. on pp. 11, 12, 14, 31, 32, 50, 104, 106, 107, 114–117, 119, 124).

[96] Yves Guiraud and Philippe Malbos, *Higher-dimensional categories with finite derivation type*, Theory Appl. Categ. 22.18 (2009), pp. 420–478 (cit. on pp. 11, 14, 15, 27, 28, 32, 37, 41, 42).

[97] Yves Guiraud and Philippe Malbos, *Identities among relations for higher-dimensional rewriting systems*, Sémin. Congr. 26 (2011), pp. 145–161 (cit. on pp. 11, 33, 46, 47).

[98] Yves Guiraud and Philippe Malbos, *Coherence in Monoidal Track Categories*, Math. Structures Comput. Sci. 22.6 (2012), pp. 931–969 (cit. on pp. 11, 33, 45).

[99] Yves Guiraud and Philippe Malbos, *Higher-dimensional normalisation strategies for acyclicity*, Adv. Math. 231.3-4 (2012), pp. 2294–2351 (cit. on pp. 11, 15, 47, 50, 52, 54, 59–61, 64, 124).

[100] Yves Guiraud and Philippe Malbos, *Polygraphs of finite derivation type*, Math. Structures Comput. Sci. 28.2 (2018), pp. 155–201 (cit. on pp. 31, 42).

[101] Yves Guiraud, Philippe Malbos, and Samuel Mimram, *A homotopical completion procedure with applications to coherence of monoids*, 24th International Conference on Rewriting Techniques and Applications (RTA 2013), vol. 21, Leibniz International Proceedings in Informatics (LIPIcs), 2013, pp. 223–238 (cit. on pp. 69, 123).

[102] Nikolai M. Günther, *Sur les modules des formes algébriques.* French, Tr. Tbilis. Mat. Inst. 9, 97-206 (1941). 1941 (cit. on p. 101).

[103] Amar Hadzihasanovic, *A combinatorial-topological shape category for polygraphs*, preprint, 2018 (cit. on p. 124).

[104] Nohra Hage and Philippe Malbos, *Knuth's coherent presentations of plactic monoids of type A*, Algebr. Represent. Theory 20.5 (2017), pp. 1259–1288 (cit. on p. 69).

[105] Lane A. Hemaspaandra, *SIGACT News Complexity Theory Column 36*, SIGACT News 33.2 (June 2002), pp. 34–47.

[106] Simon Henry, *Regular polygraphs and the Simpson conjecture*, preprint, 2018 (cit. on p. 124).

[107] Alexander Hess, *Factorable monoids: resolutions and homology via discrete Morse theory*, PhD thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2012 (cit. on p. 85).

[108] Alexander Hess and Viktoriya Ozornova, *Factorability, string rewriting and discrete Morse theory*, arXiv:1412.3025, 2014 (cit. on pp. 85–87).

[109] Anne Heyworth and Christopher D. Wensley, *Logged rewriting and identities among relators*, Groups St. Andrews 2001 in Oxford. Vol. I, vol. 304, London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 2003, pp. 256–276 (cit. on p. 32).

[110] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 (1964), pp. 205–326 (cit. on p. 101).

[111] Martin Hofmann and Thomas Streicher, *The groupoid interpretation of type theory*, Twenty-five years of constructive type theory (Venice, 1995), vol. 36, Oxford Logic Guides, Oxford Univ. Press, New York, 1998, pp. 83–111 (cit. on p. 13).

[112] Gérard Huet, *Confluent reductions: abstract properties and applications to term rewriting systems*, J. Assoc. Comput. Mach. 27.4 (1980), pp. 797–821 (cit. on p. 25).

[113] Gérard Huet, *Initiation à la théorie des catégories*, Notes du cours du DEA « Fonctionnalité, Structures de calcul et Programmation » donné l'Université Paris 7 en 1983-84 et 1984-85, 1985 (cit. on p. 33).

[114] Maurice Janet, *Sur les systèmes d'équations aux dérivées partielles*, French, J. Math. Pures Appl. 8.3 (1920), pp. 65–151 (cit. on p. 101).

[115] Matthias Jantzen, *Semi Thue systems and generalized Church-Rosser properties*, tech. rep., Bericht Nr. 92, Fachbereich Informatik, Universität Hamburg, 1982 (cit. on p. 8).

[116] Matthias Jantzen, *A note on a special one-rule semi-Thue system*, Inform. Process. Lett. 21.3 (1985), pp. 135–140 (cit. on p. 8).

[117] John F. Jardine, *A closed model structure for differential graded algebras*, Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17, Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1997, pp. 55–58 (cit. on p. 122).

[118] André Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra 175.1-3 (2002), Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 207–222 (cit. on p. 13).

[119] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. 102.1 (1993), pp. 20–78 (cit. on p. 32).

[120] Mikhail M. Kapranov and Masahico Saito, *Hidden Stasheff polytopes in algebraic* K-*theory and in the space of Morse functions*, Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), vol. 227, Contemp. Math. Providence, RI: Amer. Math. Soc., 1999, pp. 191–225 (cit. on p. 32).

[121] Mikhail Kapranov and Vladimir Voevodsky, 2-*categories and Zamolodchikov tetrahedra equations*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), vol. 56, Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 1994, pp. 177–259 (cit. on p. 68).

[122] Deepak Kapur and Paliath Narendran, *A finite Thue system with decidable word problem and without equivalent finite canonical system*, Theoret. Comput. Sci. 35.2-3 (1985), pp. 337–344 (cit. on pp. 8, 76, 123).

[123] Christian Kassel and Jean-Louis Loday, *Extensions centrales d'algèbres de Lie*, Ann. Inst. Fourier (Grenoble) 32.4 (1982), 119–142 (1983) (cit. on p. 32).

[124] David Khudaverdyan, Ashis Mandal, and Norbert Poncin, *Higher categorified algebras versus bounded homotopy algebras*, Theory Appl. Categ. 25 (2011), No. 10, 251–275 (cit. on pp. 13, 103).

[125] Jan Willem Klop, *Term rewriting systems*, Handbook of Logic in Computer Science, vol. 2, Oxford University Press, 1992, chap. 1, pp. 1–117 (cit. on p. 7).

[126] Donald Knuth and Peter Bendix, *Simple word problems in universal algebras*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Oxford: Pergamon, 1970, pp. 263–297 (cit. on pp. 25, 110, 123).

[127] Yuji Kobayashi, *Complete rewriting systems and homology of monoid algebras*, J. Pure Appl. Algebra 65.3 (1990), pp. 263–275 (cit. on pp. 49, 50, 61).

[128] Yuji Kobayashi and Friedrich Otto, *On homotopical and homological finiteness conditions for finitely presented monoids*, Internat. J. Algebra Comput. 11.3 (2001), pp. 391–403 (cit. on p. 61).

[129] Yuji Kobayashi and Friedrich Otto, *For finitely presented monoids the homological finiteness conditions FHT and bi-*FP_3 *coincide*, J. Algebra 264.2 (2003), pp. 327–341 (cit. on p. 61).

[130] Daan Krammer, *An asymmetric generalisation of Artin monoids*, Groups Complex. Cryptol. 5.2 (2013), pp. 141–167 (cit. on pp. 85, 86, 92, 95).

[131] Stephen Lack, *A Quillen model structure for 2-categories*, K-Theory 26.2 (2002), pp. 171–205 (cit. on pp. 68, 71).

[132] Stephen Lack, *A Quillen model structure for bicategories*, K-Theory 33.3 (2004), pp. 185–197 (cit. on pp. 68, 71).

[133] Yves Lafont, *A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier)*, J. Pure Appl. Algebra 98.3 (1995), pp. 229–244 (cit. on pp. 8, 10, 14, 31, 49, 61).

[134] Yves Lafont, *Equational reasoning for 2-dimensional diagrams*, Lecture Notes in Computer Science 909 (1995), pp. 170–195 (cit. on p. 31).

[135] Yves Lafont, *Towards an algebraic theory of boolean circuits*, J. Pure and Appl. Algebra 184.2-3 (2003), pp. 257–310 (cit. on pp. 8, 14, 27, 31).

[136] Yves Lafont, *Algebra and geometry of rewriting*, Appl. Categ. Structures 15.4 (2007), pp. 415–437 (cit. on p. 8).

[137] Yves Lafont and François Métayer, *Polygraphic resolutions and homology of monoids*, J. Pure Appl. Algebra 213.6 (2009), pp. 947–968 (cit. on pp. 8, 124).

[138] Yves Lafont, François Métayer, and Krzysztof Worytkiewicz, *A folk model structure on omega-cat*, Adv. Math. 224.3 (2010), pp. 1183–1231 (cit. on pp. 8, 13, 14, 50, 52).

[139] Yves Lafont and Alain Prouté, *Church-Rosser property and homology of monoids*, Math. Structures Comput. Sci. 1.3 (1991), pp. 297–326 (cit. on p. 42).

[140] Francis William Lawvere, *Functorial semantics of algebraic theories*, Reprints in Theory and Applications of Categories 5 (2004), pp. 1–121 (cit. on pp. 7, 14, 31).

[141] Jonathan Leech, *Cohomology theory for monoid congruences*, Houston J. Math. 11.2 (1985), pp. 207–223 (cit. on p. 14).

[142] Muriel Livernet, *Homotopie rationnelle des algèbres sur une opérade*, PhD thesis, Université Strasbourg 1, 1998 (cit. on p. 122).

[143] Muriel Livernet and Birgit Richter, *An interpretation of E_n-homology as functor homology*, Math. Z. 269.1–2 (2011), pp. 193–219 (cit. on p. 65).

[144] Jean-Louis Loday, *La renaissance des opérades*, Astérisque 237 (1996), Séminaire Bourbaki, Vol. 1994/95, Exp. No. 792, 3, 47–74 (cit. on p. 31).

[145] Jean-Louis Loday, *Homotopical syzygies*, Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999), vol. 265, Contemp. Math. Providence, RI: Amer. Math. Soc., 2000, pp. 99–127 (cit. on pp. 32, 51).

[146] Jean-Louis Loday and Bruno Vallette, *Algebraic Operads*, vol. 346, Grundlehren Der Mathematischen Wissenschaften, Springer, 2012 (cit. on p. 31).

[147] Maxime Lucas, *A coherence theorem for pseudonatural transformations*, J. Pure Appl. Algebra 221.5 (2017), pp. 1146–1217 (cit. on p. 33).

[148] Maxime Lucas, *A cubical Squier's theorem*, preprint, 2017 (cit. on p. 51).

[149] Maxime Lucas, *Cubical (ω, p)-categories*, preprint, 2017 (cit. on p. 51).

[150] Maxime Lucas, *Cubical categories for homotopy and rewriting*, Theses, Université Paris 7, Sorbonne Paris Cité, 2017 (cit. on p. 51).

[151] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Reprint of the 1977 edition, Berlin: Springer-Verlag, 2001, pp. xiv+339 (cit. on p. 68).

[152] Saunders Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies 49.4 (1963), pp. 28–46 (cit. on pp. 31, 32).

[153] Saunders Mac Lane, *Categorical algebra*, Bull. Amer. Math. Soc. 71 (1965), pp. 40–106 (cit. on pp. 31, 33, 43).

[154] Saunders Mac Lane, *Categories for the working mathematician*, Second, vol. 5, Graduate Texts in Mathematics, New York: Springer-Verlag, 1998, pp. xii+314 (cit. on pp. 13, 75).

[155] Francis S. Macaulay, *The algebraic theory of modular systems*, Revised reprint of the 1916 original, With an introduction by Paul Roberts, 1994 (cit. on p. 101).

[156] Georges Maltsiniotis, *Grothendieck ∞-groupoids, and still another definition of ∞-categories*, preprint, 2010 (cit. on p. 13).

[157] Andrei Markov, *On the impossibility of certain algorithms in the theory of associative systems*, Doklady Akad. Nauk SSSR (N.S.) 55 (1947), pp. 583–586 (cit. on p. 6).

[158] Andrei Markov, *On the impossibility of certain algorithms in the theory of associative systems II*, Doklady Akad. Nauk SSSR (N.S.) 58 (1947), pp. 353–356 (cit. on p. 7).

[159] Stuart McGlashan, Elton Pasku, and Stephen J. Pride, *Finiteness conditions for rewriting systems*, Internat. J. Algebra Comput. 15.1 (2005), pp. 175–205 (cit. on p. 49).

[160] François Métayer, *Resolutions by polygraphs*, Theory Appl. Categ. 11 (2003), pp. 148–184 (cit. on pp. 8, 13, 14, 50).

[161] François Métayer, *Cofibrant Objects among Higher-Dimensional Categories*, Homology, Homotopy Appl. 10.1 (2008), pp. 181–203 (cit. on pp. 8, 14, 50, 52).

[162] Yves Métivier, *About the Rewriting Systems Produced by the Knuth-Bendix Completion Algorithm*, Inf. Process. Lett. 16.1 (1983), pp. 31–34 (cit. on p. 25).

[163] Jean Michel, *A note on words in braid monoids*, J. Algebra 215.1 (1999), pp. 366–377 (cit. on p. 69).

[164] Samuel Mimram, *Computing Critical Pairs in 2-Dimensional Rewriting Systems*, Rewriting Techniques and Applications, vol. 6, Leibniz International Proceedings in Informatics (LIPIcs), Edinburgh, Royaume-Uni, 2010, pp. 227–242 (cit. on pp. 8, 14).

[165] Samuel Mimram, *Towards 3-dimensional rewriting theory [reprint of MR3194020]*, Log. Methods Comput. Sci. Special issue: Selected papers of the conference "Rewriting Techniques and Applications RTA 2010" (2014), 2:1, 47 (cit. on p. 14).

[166] Barry Mitchell, *Rings with Several Objects*, Advances in Mathematics 8 (1972), pp. 1–161 (cit. on p. 60).

[167] Teo Mora, *An introduction to commutative and noncommutative Gröbner bases*, Theoret. Comput. Sci. 134.1 (1994), Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992), pp. 131–173 (cit. on p. 102).

[168] Tadao Murata, *Petri nets: Properties, analysis and applications*, Proceedings of the IEEE 77.4 (Apr. 1989), pp. 541–580 (cit. on p. 7).

[169] Maxwell H. A. Newman, *On theories with a combinatorial definition of "equivalence"*, Ann. of Math. (2) 43.2 (1942), pp. 223–243 (cit. on pp. 7, 25).

[170] Maurice Nivat, *Congruences parfaites et quasi-parfaites*, Séminaire P. Dubreil, 25e année (1971/72), Algèbre, Fasc. 1, Exp. No. 7, Secrétariat Mathématique, Paris, 1973, p. 9 (cit. on p. 25).

[171] Friedrich Otto, Masashi Katsura, and Yuji Kobayashi, *Infinite convergent string-rewriting systems and cross-sections for finitely presented monoids*, J. Symbolic Comput. 26.5 (1998), pp. 621–648 (cit. on p. 125).

[172] Viktoriya Ozornova, *Factorability, discrete Morse theory, and a reformulation of* $K(\pi,1)$-*conjecture*, PhD thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2013 (cit. on p. 85).

[173] Luis Paris, *Artin monoids inject in their groups*, Comment. Math. Helv. 77.3 (2002), pp. 609–637 (cit. on p. 125).

[174] John Pedersen, *Morphocompletion for one-relation monoids*, RTA, vol. 355, LNCS, Springer, 1989, pp. 574–578 (cit. on p. 123).

[175] Renée Peiffer, *Über Identitäten zwischen Relationen*, Math. Ann. 121 (1949), pp. 67–99 (cit. on p. 32).

[176] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, vol. 37, University Lecture Series, Providence, RI: American Mathematical Society, 2005, pp. xii+159 (cit. on p. 115).

[177] Timothy Porter, *Crossed modules in Cat and a Brown-Spencer theorem for 2-categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 26.4 (1985), pp. 381–388 (cit. on p. 32).

[178] Timothy Porter, *Some categorical results in the theory of crossed modules in commutative algebras*, J. Algebra 109.2 (1987), pp. 415–429 (cit. on p. 32).

[179] Emil L. Post, *Recursive unsolvability of a problem of Thue*, J. Symbolic Logic 12 (1947), pp. 1–11 (cit. on p. 6).

[180] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970), pp. 39–60 (cit. on pp. 102, 104).

[181] Stephen J. Pride, *Low-dimensional homotopy theory for monoids*, Internat. J. Algebra Comput. 5.6 (1995), pp. 631–649 (cit. on pp. 10, 49, 61).

[182] The Univalent Foundations Program, *Homotopy type theory—univalent foundations of mathematics*, The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013, pp. xiv+589 (cit. on p. 13).

[183] Daniel Quillen, *Higher algebraic* K*-theory. I*, Lecture Notes in Mathematics 341 (1973), pp. 85–147 (cit. on p. 14).

[184] Kurt Reidemeister, *Über Identitäten von Relationen*, Abh. Math. Sem. Univ. Hamburg 16 (1949), pp. 114–118 (cit. on pp. 32, 51).

[185] Elsayed A. El-Rifai and Hugh R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) 45.180 (1994), pp. 479–497 (cit. on p. 85).

[186] Mark Ronan, *Lectures on buildings*, Updated and revised, Chicago, IL: University of Chicago Press, 2009, pp. xiv+228 (cit. on pp. 67, 83).

[187] Raphaël Rouquier, *2-Kac-Moody algebras*, preprint, 2008 (cit. on p. 68).

[188] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, vol. 6, Algorithms and Computation in Mathematics, Springer-Verlag, Berlin, 2000, pp. viii+254 (cit. on p. 102).

[189] Peter Selinger, *Generators and relations for* n*-qubit Clifford operators*, Log. Methods Comput. Sci. 11.2 (2015), 2:10, 17 (cit. on p. 123).

[190] Anatoly Illarionovich Shirshov, *Some algorithmic problems for Lie algebras.* Russian, Sib. Mat. Zh. 3 (1962), pp. 292–296 (cit. on p. 101).

[191] Emil Sköldberg, *Morse theory from an algebraic viewpoint*, Trans. Amer. Math. Soc. 358.1 (2006), pp. 115–129 (cit. on pp. 102, 105).

[192] Craig C. Squier, *Word problems and a homological finiteness condition for monoids*, J. Pure Appl. Algebra 49.1-2 (1987), pp. 201–217 (cit. on pp. 9, 10, 25, 42, 49, 51, 61).

[193] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi, *A finiteness condition for rewriting systems*, Theoret. Comput. Sci. 131.2 (1994), pp. 271–294 (cit. on pp. 10, 31, 36, 42, 49).

[194] James Dillon Stasheff, *Homotopy associativity of* H*-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. 108 (1963), pp. 293–312 (cit. on p. 29).

[195] Ross Street, *Limits indexed by category-valued* 2*-functors*, J. Pure Appl. Algebra 8.2 (1976), pp. 149–181 (cit. on pp. 7, 13).

[196] Ross Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra 49.3 (1987), pp. 283–335 (cit. on pp. 7, 13).

[197] Ross Street, *Higher categories, strings, cubes and simplex equations*, Appl. Categ. Structures 3.1 (1995), pp. 29–77 (cit. on p. 13).

[198] Terese, *Term rewriting systems*, vol. 55, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2003 (cit. on p. 7).

[199] Axel Thue, *Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln.* Kristiania Vidensk. Selsk, Skr. 10.10 (1914), pp. 493–524 (cit. on p. 6).

[200] Heinrich Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys. 19.1 (1908), pp. 1–118 (cit. on p. 68).

[201] Jacques Tits, *A local approach to buildings*, The geometric vein, New York: Springer, 1981, pp. 519–547 (cit. on pp. 67, 70, 83).

[202] Grigorii S. Tseitin, *Associative calculus with insoluble equivalence problem*, Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), pp. 370–371 (cit. on p. 7).

[203] Grigorii S. Tseitin, *An associative calculus with an insoluble problem of equivalence*, Trudy Mat. Inst. Steklov. 52 (1958), pp. 172–189 (cit. on p. 7).

[204] Grigorii S. Tseitin, *An associative calculus with an unsolvable problem of equivalence*, Transl., Ser. 2, Am. Math. Soc. 94 (1970), pp. 73–92 (cit. on p. 7).

[205] Victor A. Ufnarovski, *Combinatorial and asymptotic methods in algebra*, Algebra, VI, vol. 57, Encyclopaedia Math. Sci. Berlin: Springer, 1995, pp. 1–196 (cit. on p. 102).

[206] Victor A. Ufnarovski, *Introduction to noncommutative Gröbner bases theory*, Gröbner bases and applications (Linz, 1998), vol. 251, London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 1998, pp. 259–280 (cit. on p. 102).

[207] Henry Whitehead, *Combinatorial homotopy I*, Bulletin of the American Mathematical Society 55.3 (1949), pp. 213–245 (cit. on p. 32).

[208] Henry Whitehead, *Combinatorial homotopy II*, Bulletin of the American Mathematical Society 55.5 (1949), pp. 453–496 (cit. on p. 32).